THEORETICAL EXPECTATION VALUE OF THE CAPACITY OF FUZZY POLYNOMIAL BIDIRECTIONAL HETERO-CORRELATOR

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ABSTRACT

A method of fuzzy data recall using polynomial bidirectional hetero-correlator is presented. This has a higher capacity for pattern pair storage than that of the conventional BAMs and fuzzy memories. In addition, a new energy function is defined. The polynomial bidirectional hetero-correlator (**PBHC**) takes advantage of fuzzy characteristics in evolution equations such that the signal-noise-ratio (SNR) of data recall is significantly increased. The energy of the polynomial bidirectional hetero-correlator defined by the proposed energy function decreases as the recall process proceeds, ensuring the stability of the system. The increase of SNR consequently enhances the capacity of the polynomial bidirectional hetero-correlator. Theoretical expectation value of the capacity of fuzzy data recall using polynomial bidirectional hetero-correlator is also estimated.

1. INTRODUCTION

Associative memories have been an important research area in the neural networks [1], [2]. In related works, Kosko presented a fuzzy associative memory (FAM) system structure [3]. However, no energy function introduced in his works could guarantee that every stored pattern pair resides at a local minimum on energy surfaces. Moreover, no capacity analysis was performed as well. We propose an energy function and verify that every stored pattern pair will exist at a local minimum in an energy surface for fuzzy data recall using polynomial bidirectional hetero-correlator (**PBHC**) in which the component of a fuzzy vector is termed a fuzzy bit (**fit**). The PBHC has a higher capacity for pattern pair storage than that of the conventional BAMs and fuzzy memories. In this work, we adopt the signal-noise-ratio (SNR) approach and derive the minimal Z, which is deemed as the power of the polynomial, as well as capacity of the PBHC in an average case. The expectation value of the capacity of the PBHC is then attained according to an SNR analysis scheme.

2. FRAMEWORK OF HIGH CAPACITY PBHC

2.1 Evolution Equations

Assume that we are given M pattern pairs, which are $\{(X_1,Y_1),(X_2,Y_2),\ldots,(X_M,Y_M)\}$, where $X_i = (x_{i1},x_{i2},\ldots,x_{in})$, $Y_i = (y_{i1},y_{i2},\ldots,y_{ip})$. Let $1 \le i \le M$, $x_{ij} \in [0,1]$, $1 \le j \le n$, $y_{ij} \in [0,1]$, $1 \le j \le p$, n and p are the component dimensions of X and Y, and n is assumed to be smaller than or equal to p without any loss of generality. $x_{ik}, y_{ik} \in \{0/\lambda, 1/\lambda, \ldots, \lambda/\lambda\}$, fuzzy space = [1,0], λ is a fuzzy quantum, and σ is a fuzzy quantum gap. By assuming that $\lambda = 10$, $\sigma = 1/\lambda = 0.1$ and $1/(2\lambda) = \sigma/2 = 0.05$ can be obtained. Instead of using Kosko's approach, we use the following evolution equations in the recall process of the PBHC

$$y_k = \frac{\sum_{i=1}^{M} y_{ik} \cdot \left(\frac{u - \|X_i - X\|^2}{u} \right)^{M^Z}}{\sum_{i=1}^{M} \left(\frac{u - \|X_i - X\|^2}{u} \right)^{M^Z}}, \tag{1}$$

$$x_k = \frac{\sum_{i=1}^{M} x_{ik} \cdot \left(\frac{u - \|Y_i - Y\|^2}{u} \right)^{M^Z}}{\sum_{i=1}^{M} \left(\frac{u - \|Y_i - Y\|^2}{u} \right)^{M^Z}} \tag{2}$$

where M denotes the number of patterns in the PBHC, X_i, Y_i, $i = 1,\ldots,M$, represent the stored patterns, X or Y is the initial vector presented to the network, x_k and x_{ik} denote the kth digits of X and X_i, respectively, y_k and y_{ik} represent the kth digits of Y and Y_i, respectively, Z is a positive integer, and u denotes a function defined as the following equation:

$$u = \sum_{i=1}^{M} \sum_{j=1}^{M} \left(\|X_i - X_j\|^2 + \|Y_i - Y_j\|^2 \right). \tag{3}$$

Notably, u is bounded.

2.2 Energy Function and Stability

The fact that every stored pattern pair should produce a local minimum on the energy surface to be recalled correctly accounts for why the energy function is intuitively defined as

$$E(X,Y) = \sum_{i=1}^{M} \|X - X_i\|^2 \cdot \|Y - Y_i\|^2. \tag{4}$$

Fuzzy data model using PBHC can be viewed as one kind of BAM, i.e. bidirectional associative memory. Our earlier work proposed a fuzzy data recall using a PBHC [4], which has been verified by a two-phase approach to be stable.

2.3 Analysis of the Capacity of PBHC

The SNR approach is adopted herein to compute the capacity of the PBHC. Let X_i and Y_i be the stored pattern pairs. Assume that X_1 is the input pattern pair and Y_1 is recalled expectantly. Substituting X_1 for X allows us to rewrite Eqn.(1) as

$$y_k \cdot \sum_{i=1}^{M} \left(\frac{u - \|X_i - X_1\|^2}{u} \right)^{M^Z}$$
$$= y_{1k} \cdot 1^2 + y_{2k} \cdot \left(\frac{u - \|X_2 - X_1\|^2}{u} \right)^{M^Z}$$
$$+ y_{3k} \cdot \left(\frac{u - \|X_3 - X_1\|^2}{u} \right)^{M^Z}$$
$$+ \ldots + y_{Mk} \cdot \left(\frac{u - \|X_M - X_1\|^2}{u} \right)^{M^Z} \tag{5}$$

The first term in the above equation corresponds to the signal, and the other terms are the noise. The power of signal is

$$S = y_{1k} \cdot 1^2. \tag{6}$$

Besides the first term, the remaining terms are actually the sum of M-1 independent identically distributed random variables. Therefore, the noise of these terms is M-1 times of the noise of a single random variable. Let

$$v_2 = y_{2k} \cdot \left(\frac{u - \|X_2 - X_1\|^2}{u} \right)^{M^Z}$$
$$v_3 = y_{3k} \cdot \left(\frac{u - \|X_3 - X_1\|^2}{u} \right)^{M^Z}$$
$$\vdots$$
$$v_M = y_{Mk} \cdot \left(\frac{u - \|X_M - X_1\|^2}{u} \right)^{M^Z}.$$

Since all of the v_i's, i=2 to M, have the same property, we select v_2 as the sample. By assuming that $X_1=(x_{11},x_{12},\ldots,x_{1n})$, $X_2=(x_{21},x_{22},\ldots,x_{2n})$, then we can obtain

$$\Delta = \|x_{21} - x_{11}\| \in \{0/\lambda, 1/\lambda, \ldots, 1/2, \ldots, \lambda/\lambda\} \tag{7}$$

Also assume that Δ is the difference of a fuzzy bit (fit). It is trivial to derive the following general form of probability function for the difference of a fuzzy bit (fit)

$$P_{fit}\left(\Delta = \frac{i}{\lambda}\right) = \frac{2\cdot(\lambda - i + 1)}{(\lambda+1)^2} \tag{8}$$

Where $1 \le i \le \lambda$. In addition, we can derive the expectation value for the difference of a fit as follows,

$$E(\Delta x_{i1}) = \sum_{i=1}^{\lambda} \frac{i}{\lambda} \cdot \frac{2\cdot(\lambda - i + 1)}{(\lambda+1)^2} = \frac{(\lambda+2)}{3(\lambda+1)} \tag{9}$$

The expectation value of the square of the difference of a fit can also be derived as follows,

$$E(\Delta x_{i1}^2) = \sum_{i=1}^{\lambda} \left(\frac{i}{\lambda}\right)^2 \cdot \frac{2(\lambda - i + 1)}{(\lambda+1)^2} = \frac{\lambda+2}{6\lambda}. \tag{10}$$

The mean of one noise term can be derived as

$$E(v_i) = \left(\frac{u - \|X_i - X\|^2}{u} \right)^{M^Z} \cdot E(y_{ik})$$
$$= \left(\frac{u - \|X_i - X\|^2}{u} \right)^{M^Z} \cdot \sum_{i=0}^{\lambda} \frac{i}{\lambda} \cdot \frac{1}{\lambda+1}$$
$$= \frac{1}{2} \left(\frac{u - \|X_i - X\|^2}{u} \right)^{M^Z} \tag{11}$$

The expectation value of the power of one noise term can be derived as

$$E(v_i^2) = \left[\left(\frac{u - \|X_i - X\|^2}{u} \right)^{M^Z} \right]^2 \cdot E(y_{ik}^2)$$
$$= \left[\left(\frac{u - \|X_i - X\|^2}{u} \right)^{M^Z} \right]^2 \cdot \sum_{i=0}^{\lambda} \frac{1}{\lambda+1} \left(\frac{i}{\lambda}\right)^2$$
$$= \left[\left(\frac{u - \|X_i - X\|^2}{u} \right)^{M^Z} \right]^2 \cdot \frac{2\lambda+1}{6\lambda}. \tag{12}$$

From the above Eqn.(3) and Eqn.(10), the expectation value of u can be obtained as follows,

$$E(u) = C_2^M (n+p) \cdot \frac{\lambda+2}{6\lambda}. \tag{13}$$

The SNR (signal-noise-ratio) must be greater than one in order to

$$SNR = \frac{Signal\ Power}{total\ Noise\ Power} = \frac{S}{(M-1)\cdot Noise}$$
$$= \frac{Signal}{(M-1)\cdot(Noise\ term's\ Variance)} > 1. \tag{14}$$

recall the correct pattern pair

The variance of noise can be derived as

$$Var(v_i) = E(v_i^2) - E^2(v_i) \tag{15}$$

since

$$E(v_i^2) > E(v_i^2) - E^2(v_i). \tag{16}$$

Thus, the following inequality can be obtained:

$$E(v_i^2) > Var(v_i). \tag{17}$$

The above upper bound in Eqn.(17) is the maximal noise power, called N_{max}, which is equal to $E(v_i^2)$. Then, the minimal signal-noise-ratio (SNR_{min}) of the PBHC is

$$SNR_{min} = \frac{S}{(M-1)N_{max}}$$

$$= \frac{1}{(M-1)\cdot\left\{\left[\left(u - n((1/3)\cdot(\lambda+2)/(\lambda+1))^2\right)/u\right]^{1/Z}\right\}^2 \cdot \frac{2\lambda+1}{6\lambda}} > 1 \quad (18)$$

Next, the minimal Z in the average case for PBHC is derived to accurately recall every stored pattern pair according to Eqn.(18) as follows,

$$Z > \frac{1}{ln\, M} \cdot ln\left\{\frac{(1/2)ln[6\lambda/((M-1)\cdot(2\lambda+1))]}{ln[(u - n((1/3)\cdot(\lambda+2)/(\lambda+1))^2)/u]}\right\}. \quad (19)$$

where u can be replaced with $E(u) = C_2^M(n+p)\cdot\frac{\lambda+2}{6\lambda} = \frac{M(M-1)}{2}\cdot(n+p)\cdot\frac{\lambda+2}{6\lambda}$ to derive the expectation value of Z's lower bound. The above Eqn.(19) is the accurate solution of Z. According to Eqn.(18), the capacity can also be derived as follows,

$$M > \left\{\frac{(1/2)ln[6\lambda/((M-1)\cdot(2\lambda+1))]}{ln[u - n((1/3)\cdot(\lambda+2)/(\lambda+1))^2)/u]}\right\}^{Z^{-1}}.$$

$$= \left\{\frac{(1/2)ln[6\lambda/((M-1)\cdot(2\lambda+1))]}{ln[1 - [(1/3)\cdot(\lambda+2)/(\lambda+1)]^2/[M\cdot(M-1)\cdot(\lambda+2)/6\lambda]]}\right\}^{Z^{-1}} \quad (20)$$

where u can again be substituted with $E(u)$. According to Eqn.(20), if $n = p$ is assumed, we can obtain the capacity, M, which is irrelevant to n. Thus, this observation implies that the theoretical expectation value of the capacity of the fuzzy PBHC should be the largest possible combinations given by $\{0/\lambda, 1/\lambda, \ldots, \lambda/\lambda\}^n$ as follows,

$$M = (1+\lambda)^n. \quad (21)$$

Moreover, we assume $\lambda >> 1$ and $n = p$ to simplify $E(u)$ as follows,

$$\begin{aligned} E(u) &= \frac{M(M-1)}{2}\cdot(n+p)\cdot\frac{\lambda+2}{6\lambda} \\ &\geq \frac{M(M-1)}{2}\cdot(2n)\cdot\frac{\lambda+2}{6\lambda} \\ &\cong \frac{M(M-1)n}{6}. \end{aligned} \quad (22)$$

Thus, an approximate solution for Z can be obtained as follows,

$$Z > \frac{1}{ln\, M} \cdot ln\left\{\frac{(1/2)ln[3/(M-1)]}{ln[(M(M-1)-(2/3))/(M(M-1))]}\right\}. \quad (23)$$

3. SIMULATION ANALYSIS

To verify the capacity analysis, we utilized computer programs accurate solution and the approximate solution of Z, respectively. Fig. 1 and Fig. 2 plot the above results, respectively. Fig. 3 and Fig. 4 compare the results of M vs. Z in the accurate solution and the approximate solution of Z, where λ is equal to 10 and 2, respectively (The legends of $Z(1)$ and $Z(2)$ represent the accurate solution and the approximate solution of Z, respectively). Fig. 5 summarizes the result of capacity, M, vs. n given $\lambda = 10$. Because the numerical values of the capacity are too large, a log scale is used such that the contrast becomes clearer. According to this figure, the fuzzy data recall using the PBHC in the average case provides a significantly high capacity of storage for patterns.

Example 1. To verify the capacity, M, described in Eqn.(21) of Section 2.3 is correct, we use the PBHC to store and recall a set of 10,000 ($\lambda = 9$, $n = 4$) fuzzy patterns which form 5,000 different pattern pairs. In this simulation, all of the pattern pairs are randomly generated; in addition, these patterns are all unique. By assuming that $\lambda = 9$ and $n = 4$, then capacity $M = 10{,}000$ according to Eqn.(21). For instance, if the network stores the patterns as

$$(X_1, Y_1) = ((0.5, 0.7, 0.2, 0.9), (0.6, 0.2, 0.5, 0.1))$$
$$(X_2, Y_2) = ((0.3, 0.4, 0.1, 0.6), (0.4, 0.6, 0.1, 0.8))$$
$$\vdots$$
$$(X_{5000}, Y_{5000}) = ((0.8, 0.2, 0.3, 0.7), (0.9, 0.2, 0.5, 0.1))$$

Simulation results indicate that just one iteration is required for every X_i (pattern pair) to recall its Y_i (pattern pair) correctly, and vice versa.

For the sake of correctness, the simulation is also performed for the other cases, which include $M = 1296$ ($\lambda = 5$, $n = 4$) and $M = 1{,}000$ ($\lambda = 9$, $n = 3$). Those results also verify the validity of recalling stored pattern pairs.

4. CONCLUSION

According to our results, the fuzzy data recall using PBHC provides an extremely high storage capacity for patterns. This method utilizes a fuzzy scheme to magnify the SNR. The proposed energy function ensures that every stored pattern pair is located in a local minimum of the energy surface. The capacity of the PBHC in the average case is estimated, thereby allowing us to predetermine the size of the PBHC by the demand of capacity possible.

5. REFERENCES

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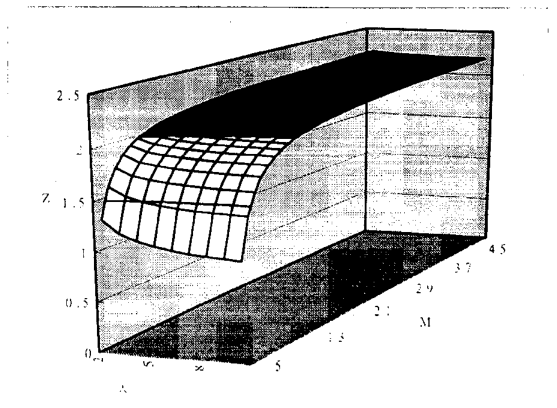

Fig. 1: The relationship of M, Z and λ for the accurate solution of Z ($n = 2$)

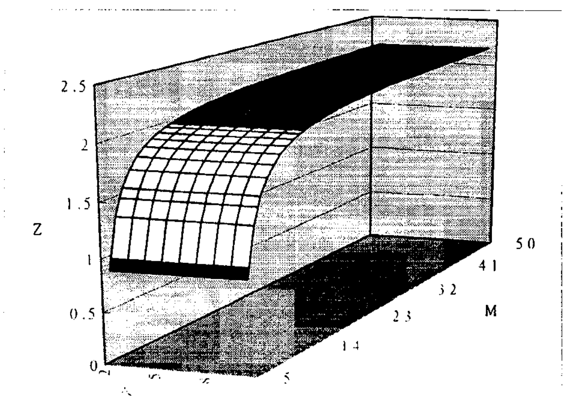

Fig. 2: The relationship of M, Z and λ for approximate solution of Z ($n = 2$)

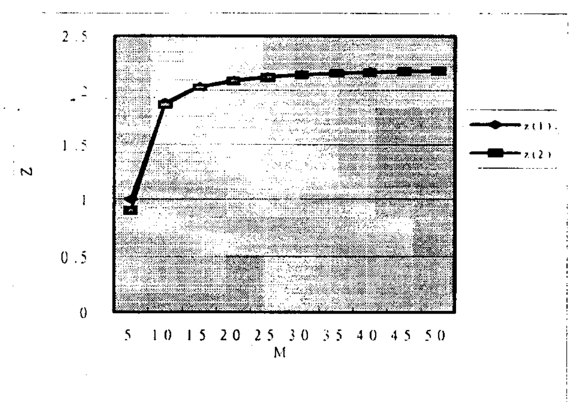

Fig. 3: The comparison of M vs. Z in the accurate solution and the approximate solution ($\lambda = 10$)

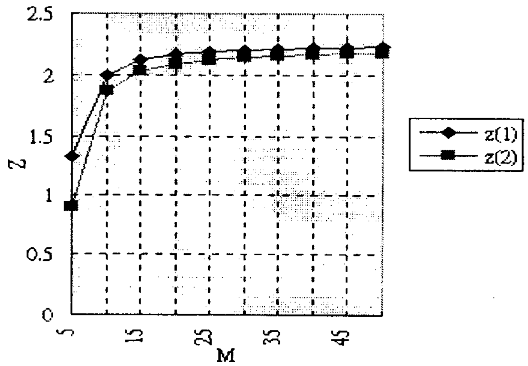

Fig. 4: The comparison of M vs. Z in the accurate solution and the approximate solution ($\lambda = 2$)

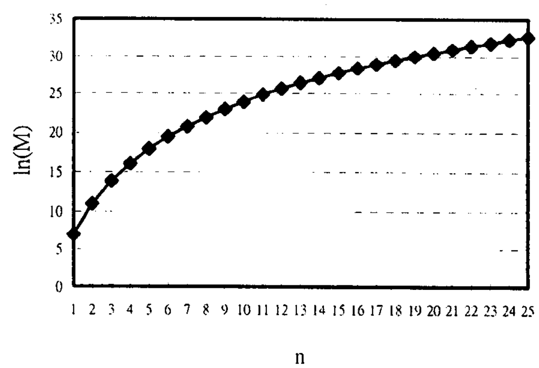

Fig. 5: The capacity of the PBHC in the average case ($\lambda = 10$)