The Decision-Making Properties of Discrete Multiple Exponential Bidirectional Associative Memories

Chua-Chin Wang and Jyh-Ping Lee

***Abstract*— A method for modeling the learning of belief combination in evidential reasoning using a neural network is presented. A centralized network composed of multiple exponential bidirectional associative memories (eBAM's) sharing a single output array of neurons is proposed to process the uncertainty management of many pieces of evidence simultaneously. The stability of the proposed multiple eBAM network is proved. The sufficient condition to recall a stored pattern pair is discussed. Most important of all, a majority rule of decision making in presentation of multiple evidence is also found by the study of signal-noise-ratio of multiple eBAM network. A guaranteed stable state condition, i.e., the condition for the fastest recall of a pattern pair, is also studied. The result is coherent with the intuition of reasoning.**

I. Introduction

NEURAL networks have been drawing increasing interest as powerful tools to solve different tasks of artificial intelligence [2], [3], [12]. An associative memory is one type of neural network which essentially is a single functional layer or slab that associates one set of vectors with another set of vectors. Kosko [6] proposed a two-level nonlinear network, bidirectional associative memory (BAM), which extends a one-directional process to a two-directional process. Jeng [5] and Wang [13], respectively, then generalized the concept of storing information in the exponential BAM (eBAM).

Among the problems of evidential reasoning, conflicts caused by sequential programming and partial dependency are pretty hard to be fully resolved [8], [10]. The basic reason is all of the traditional methods for evidential reasoning are developed for two pieces of evidence. Thus, when there are more than two pieces of evidence, conflicts will happen if the combination orders are different [11]. Wang *et al.* pointed out the importance of simultaneously processing many pieces of evidence [10], and he further proposed a method using multiple BAM structure to handle the demand of combining many evidence at the same time [12]. Because the relationship of evidence and the hypothesis is always referred to be an IF-and-THEN relationship. Hence, this IF-and-THEN format can be easily transformed into numbers which can be stored in memories, more specifically, associative memories. If people intend to evaluate the degree of a piece of evidence supporting a hypothesis, they simply present the evidence to their memory to recall stored information. If there are more than a piece of evidence, then present all of the evidence and see the result of their common output. Since the more evidence support one hypothesis, the result should be drawn closer to this hypothesis. Due to the inherently poor capacity of BAM [5], [13], however, obviously the multiple BAM network would be limited to a foreseeable degree of processing capability. We propose a multiple eBAM network to increase the processing capability of reasoning many evidence. We also discuss the majority rule of decision making for handling many evidence at the same time. The majority rule means that if more than half of the presented evidence support one hypothesis, the result of the belief combination of all of the evidence must be inclined toward this hypothesis. This rule is intuitively in accordance with the human reasoning.

In this paper, we adopt the exponential form and combine it with the multiple BAM structure to enhance the signal-noise-ratio (SNR) of the entire network and, consequently, increase the capacity. We also prove the stability of the multiple eBAM network. A majority factor (k) is determined to indicate what the portion of the total amount of eBAM is necessary to reach the dominant hypothesis. The guaranteed stable state condition of pattern pairs, or the fastest recall condition of pattern pairs, is studied. The simulation result is much more appealing than the previous works.

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The authors are with the Department of Electrical Engineering, National Sun Yat-Sen University, Kaohsiung, Taiwan 80424.

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II. Framework of Multiple Exponential BAM'S Network

A. Evolution Equations

As shown in Fig. 1, the multi-eBAM network is constructed with L single eBAM's which share a common output array of neurons. In each clock, the input vectors are presented at the input array of neurons, respectively.

Suppose we are given N training sample pairs to the qth eBAM of the network, which are

$$\{(A_{q1}, B_1), (A_{q2}, B_2), \cdots, (A_{qN}, B_N)\} \tag{1}$$

where

$$A_{qi} = (a_{qi1}, a_{qi2}, \cdots, a_{qin})$$
$$B_i = (b_{i1}, b_{i2}, \cdots, b_{ip}).$$

Let X_i and Y_i be the bipolar mode of the training pattern pairs, A_{qi} and B_i, respectively. That is, $X_{qi} \in \{-1, 1\}^n$ and $Y_i \in \{-1, 1\}^p$. Thus, we use the following evolution equations

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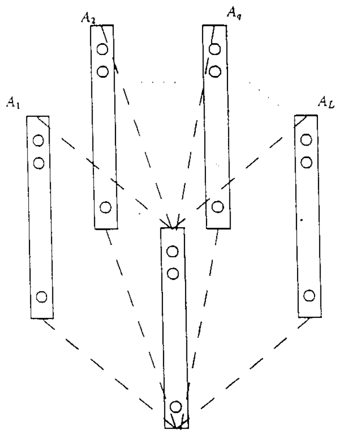

Fig. 1. The configuration of a multi-eBAM neural network.

in the recall process of the multi-eBAM network

$$y_k = \begin{cases} 1 & \text{if } \sum_{q=1}^{L} \sum_{i=1}^{N} y_{ik} b^{X_{qi}\cdot X_q} \ge 0 \\ -1 & \text{if } \sum_{q=1}^{L} \sum_{i=1}^{N} y_{ik} b^{X_{qi}\cdot X_q} < 0 \end{cases} \quad (2)$$

$$x_{qk} = \begin{cases} 1, & \text{if } \sum_{i=1}^{N} x_{qik} b^{Y_i\cdot Y} \ge 0 \\ -1, & \text{if } \sum_{i=1}^{N} x_{qik} b^{Y_i\cdot Y} < 0 \end{cases}$$

where b is a positive number, $b > 1$ "$\cdot$" represents the inner product operator, x_{qk} and x_{qik} are the kth bits of X_q and the X_{qi}, respectively, and y_k and y_{ik} are for Y and the Y_i, respectively.

B. Energy function and Stability

Since every stored pattern pair should produce a local minimum on the energy surface [13], the energy function is intuitively defined as

$$E(X,Y) = -\sum_{i=1}^{N} b^{X_i\cdot X} - \sum_{i=1}^{N} b^{Y_i\cdot Y}. \quad (3)$$

Thus, the multi-eBAM network's overall energy function is defined as

$$E = \sum_{q} E(X_q, Y) = -\sum_{q=1}^{L}\sum_{i=1}^{N} (b^{X_{qi}\cdot X_q} + b^{Y_i\cdot Y}). \quad (4)$$

Assume $E(X_q^{'}, Y)$ is the energy of next state in which Y stays the same as in the previous state, and all of the other eBAM's stay at the same state as before. Hence, $\Delta E_{xq} = -\sum_{i=1}^{N} b^{X_{qi}\cdot X_q^{'}} - (-\sum_{i=1}^{N} b^{X_{qi}\cdot X_q})$. Assume the ith pair is the target of the recall process for the qth eBAM. Let d_{qxi} be the Hamming distance between X_q and X_{qi}, $d_{qxi}^{'}$ the Hamming distance between the $X_q^{'}$ and X_{qi}. Hence the ΔE_{xq} can be modified to be

$$\Delta E_{xq}^{'} = -\sum_{i=1}^{N} \log_b(b^{n-2d_{qxi}^{'}}) + \sum_{i=1}^{N} \log_b(b^{n-2d_{qxi}}) = -\sum_{i=1}^{N}\sum_{k=1}^{n} (x_{qk}^{'} - x_{qk}) x_{qik}. \quad (5)$$

Note that log is used, which is a monotonic function. From the recall process shown by (3) and (5), the $\Delta E_{xq} < 0$ is ensured. A similar result was also given in (5) of Jeng *et al.* [5]. Therefore, according to Jeng's conclusion, (3) makes $(x_{qk}^{'} - x_{qk})x_{qik}$ always nonnegative such that $\Delta E_{xq}^{'} \le 0$, and

$$\Delta E_{xq}^{'} \le 0 \Rightarrow -\sum_{i=1}^{N} \log_b(b^{n-2d_{qxi}^{'}}) \le -\sum_{i=1}^{N} log_b(b^{n-2d_{qxi}})$$
$$\Rightarrow -\sum_{i=1}^{N} b^{X_{qi}\cdot X_q^{'}} \le -\sum_{i=1}^{N} b^{X_{qi}\cdot X_q}$$
$$\Rightarrow \Delta E_{xq} \le 0.$$

Obviously, it also holds for the other case $E(X_q, Y^{'}) \le E(X_q, Y)$ if the pair is heading for a stored pair, (X_{qi}, Y_i). Since the $E(X_q, Y)$ is bounded by $-N(b^n + b^p) \le E(X,Y) \le -N(b^{-n} + b^{-p})$ for all X_q's and Y, the energy of the exponential BAM will converge to a stable local minimum.

C. Sufficient Conditions to Recall a Stored Pattern Pair

The requirement for recalling a pair was suggested by Kosko [6], stating that a pattern pair must have the local minimal energy. As to a single eBAM, suppose the pattern pair (X_{qi}, Y_i), $i = 1, 2, \cdots, N$ are the stored pairs. Let the Hamming distance between X and X_{qi} be one, which denotes the distance from the closest pattern pairs. Thus, we conclude the criteria ensured recall are

$$-b^{X_{qi}\cdot X} - b^{Y_i\cdot Y_i} \ge -b^{X_{qi}\cdot X_{qi}} - b^{Y_i\cdot Y_i}$$
$$-b^{n-2} \ge -b^{n}.$$

It must be true. We can make sure that the eBAM must be guaranteed to have the correct recall according to Kosko's criteria.

D. A Majority Rule for the Multi-eBAM Network

1) The Majority Rule of a Special Case: According to the discussion in the previous sections, every single eBAM tends to store their own pattern pairs in the local minimums of their network, respectively. Assume there are L single eBAM consisting of a multiple eBAM network, and these eBAM's share a single output array of processing units. If these individual eBAM's are activated by respective input patterns and they do not "agree" to have the same conclusion, i.e., the same output pattern, what will be the final result of the whole network? This problem is like a reasoning mechanism which takes many evidence into consideration at the same time to reach an optimal estimation of the hypothesis.

Hence, we formulate the entire problem as follows: Given a multi-eBAM network composed of L single eBAM's, what is the minimal majority factor k, $k \in [0,1]$, to make kL eBAM's, which are vowing a common output pattern and the other eBAM's are not, dominate the common output? In other words, we are interested in exploring the lower bound of the kL which can force the output pattern to be their common output pattern. Note that in fact the kL denotes an integer, $Ceiling(kL)$, which is the smallest integer larger than kL. In the following text, we simply use the kL without any loss of robustness.

Before we discuss the lower bound of kL, we have to study an extreme case in which a upper bound of kL will be derived. Assume the pattern pairs, $(X_{11}, Y_r), (X_{21}, Y_r), \cdots, (X_{kL1}, Y_r)$, are encoded in 1st to kLth eBAM's, respectively, and pattern pairs, $(X_{(kL+1)1}, Y_s), \cdots, (X_{L1}, Y_s)$, are stored in $(kL+1)$th to Lth eBAM's, respectively. Thus, when input patterns, $X_{11}, X_{21}, \cdots, X_{L1}$, are presented at the input array of each individual eBAM, what would be the result of output?

Suppose the Y_r is the output pattern that we are looking for, i.e., it is deemed as the signal. By the SNR approach [1], [5], [13] and the evolution equations (3), we are aware of the following facts

$$\begin{aligned}\sum_{q=1}^{L}\sum_{i=1}^{N} y_{ij} b^{X_{qi}\cdot X_q} &= \sum_{q=1}^{kL}\sum_{i=1}^{N} y_{ij} b^{X_{qi}\cdot X_{qr}} \\ &\quad + \sum_{q=kL+1}^{L}\sum_{i=1}^{N} y_{ij} b^{X_{qi}\cdot X_{qs}} \\ &= \sum_{q=1}^{kL}\Big(y_{rj}b^n + \sum_{i\neq r} y_{ij} b^{X_{qi}\cdot X_{qr}}\Big) \\ &\quad + \sum_{q=kL+1}^{L}\Big(y_{sj}b^n + \sum_{i\neq s} y_{ij} b^{X_{qi}\cdot X_{qs}}\Big)\end{aligned}$$

where X_q represents the input pattern presented to the qth eBAM. The X_q can be shown in detail as follows

$$X_q = \begin{cases} X_{1r}, \cdots, X_{kLr} & \text{to} \quad 1\text{st}, \cdots, kL\text{theBAM, respectively} \\ X_{(kL+1)r}, \cdots, X_{Lr} & \text{to} \quad (kL+1)\text{th}, \cdots, L\text{theBAM, respectively.} \end{cases}$$

By our previous assumptions, only the first term in the above equation is the signal we wish to observe at the output array of processing units. As for the rest terms, they are the undesired noise. Therefore, we can derive the signal power as

$$\begin{aligned} S &= \sum_{q=1}^{kL} b^{2n} \\ &= kLb^{2n} \end{aligned}$$

and the largest power of noise, which means all of the rest $(1-k)L$ eBAM's support another output pattern Y_s, is

$$\begin{aligned} N &= (1-k)Lb^{2n} + kL(N-1)b^{2(n-2)} \\ &\quad + (1-k)L(N-1)b^{2(n-2)} \\ &= (1-k)Lb^{2n} + L(N-1)b^{2(n-2)}. \end{aligned}$$

In the above noise power equation, we assume not only all of the rest $(1-k)L$ eBAM's support another output pattern, but also this pattern is the closest pattern to the desired one. If the desired output is intended to be recalled, then the sufficient condition is the $S > N$ according to the SNR approach. Thus we can conclude the lower bonds for this definite recall condition of k is

$$kLb^{2n} > (1-k)Lb^{2n} + L(N-1)b^{2(n-2)} \tag{6}$$

$$k > \frac{1}{2} + \frac{N-1}{2b^4}. \tag{7}$$

Note that this lower bound of k means any k bigger than this threshold can force the output pattern to be the common desired output pattern of the kL eBAM's in the network. If the bound of (7) is larger than one, however, it means even all of the eBAM's support one output pattern, there is no guarantee to recall this common pattern.

2) The Majority Rule of the General Case: In the above extreme case, we assume all of the rest $(1-k)L$ eBAM's support another same output pattern which is only one bit Hamming distance away from the desired pattern. Generally speaking, however, most of the reasoning problems will not be this special. We will consider a general case in which kL eBAM's still support a common output pattern, but the rest $(1-k)L$ eBAM's do not support the same output pattern, i.e., they individually support their own output patterns, respectively. Basing upon this assumption, then we can derive the following results

$$\begin{aligned}\sum_{q=1}^{L}\sum_{i=1}^{N} y_{ij} b^{X_{qi}\cdot X_q} &= \sum_{q=1}^{kL}\sum_{i=1}^{N} y_{ij} b^{X_{qi}\cdot X_{qr}} \\ &\quad + \sum_{q=kL+1}^{L}\sum_{i=1}^{N} y_{ij} b^{X_{qi}\cdot X_{qs}} \\ &= \sum_{q=1}^{kL}\Big(y_{rj}b^n + \sum_{i\neq r} y_{ij} b^{X_{qi}\cdot X_{qr}}\Big) \\ &\quad + \sum_{q=kL+1}^{L}\Big(y_{sj}b^n + \sum_{i\neq s} y_{ij} b^{X_{qi}\cdot X_{qs}}\Big) \\ &= kL\cdot y_{rj}b^n + (1-k)L\cdot y_{sj}b^n \\ &\quad + \sum_{q=1}^{kL}\sum_{i\neq r} y_{ij} b^{X_{qi}\cdot X_{qr}} \\ &\quad + \sum_{q=kL+1}^{L}\sum_{i\neq s} y_{ij} b^{X_{qi}\cdot X_{qs}} \end{aligned} \tag{8}$$

where X_q represents the input pattern presented to the qth eBAM. The X_q can be shown in detail as follows

$$X_q = \begin{cases} X_{1r}, \cdots, X_{kLr} & \text{to} \quad 1\text{st}, \cdots, kL\text{theBAM, respectively} \\ X_{(kL+1)r}, \cdots, X_{Lr} & \text{to} \quad (kL+1)\text{th}, \cdots, L\text{theBAM, respectively.} \end{cases}$$

The third and fourth terms of (8) can be analyzed by the SNR approach proposed by Wang [13]. The third and fourth terms are actually sums of $kL(N-1)$ and $(1-k)L(N-1)$

independent identically distributed random variables, respectively. Therefore, the variances of the third and fourth terms are $kL(N-1)$ and $(1-k)L(N-1)$ times the variance of a single random variable. Let

$$v_1 = y_{1j} b^{X_{qi} \cdot X_q}$$
$$v_2 = y_{2j} b^{X_{qi} \cdot X_q}$$
$$\vdots$$
$$v_N = y_{Nj} b^{X_{qi} \cdot X_q}.$$

Since all of the v_i's have the same property, we select v_1 as the sample. It is trivial to derive the following probability functions for v_1

$$Pr(v_1 = b^{n-2-2k}) = (\frac{1}{2})^{n-1}\, C_k^{n-1} \tag{9}$$

$$Pr(v_1 = -b^{n-2-2k}) = (\frac{1}{2})^{n-1}\, C_k^{n-1} \tag{10}$$

where the k is the Hamming distance between X_q and X_{qi}. The mean of the noise term is obviously zero. Then the variance can be derived as

$$\begin{aligned}
E[v_1^2] &= 2\sum_{k=0}^{n-1} b^{2(n-2k-2)}(\frac{1}{2})^{n-1} C_k^{n-1} \\
&= 2\sum_{k=0}^{m} b^{2(m-2k-1)}(\frac{1}{2})^{m} C_k^{m}, \quad \text{where} \quad m = n-1 \\
&= 2(\frac{1}{2})^m b^{2(m-1)} \sum_{k=0}^{m} (b^{-4})^k 1^{(m-k)} C_k^m \\
&= 2(\frac{1}{2})^m b^{-2}(b^2 + b^{-2})^m \\
&= 2(\frac{1}{2})^{n-1} \frac{(b^2 + b^{-2})^{n-1}}{b^2} \\
&= 2(\frac{1}{2})^{n-1} \frac{b^{2n}(1 + b^{-4})^{n-1}}{b^4}.
\end{aligned}$$

Hence, the power of the third and fourth terms are, respectively,

$$N_3 = E(v_1^2) \cdot kL(N-1)$$
$$N_4 = E(v_1^2) \cdot (1-k)L(N-1).$$

The SNR of this case can be further derived

$$\begin{aligned}
SNR &= \frac{kL \cdot b^{2n}}{(1-k)L \cdot b^{2n} + N_3 + N_4} \\
&= \frac{k}{(1-k) + (N-1) \cdot 2(\frac{1}{2})^{n-1} \frac{(1+b^{-4})^{n-1}}{b^4}}.
\end{aligned} \tag{11}$$

If the common desired output pattern must be recalled, then the sufficient condition is the SNR must be greater than one. Thus, we can find the lower bound of k

$$\begin{aligned}
&SNR > 1 \\
&k > (1-k) + 2^{-n+1} \cdot \frac{2(N-1)(1+b^{-4})^{n-1}}{b^4} \\
&\quad = (1-k) + \frac{2(N-1)(1+b^{-4})^{n-1}}{2^{n-1} b^4} \\
&\quad = (1-k) + \frac{1}{SNR_{eBAM}}
\end{aligned}$$

where $SNR_{eBAM} = \frac{2^{n-1} b^4}{2(N-1)(1+b^{-4})^{n-1}}$ according to Wang's analysis [13].

Therefore, we conclude the above discussion of a majority rule of multi-eBAM network with the following theorem

Theorem of the Majority Rule for a Multi-eBAM Network: Given a multi-eBAM network with L single eBAM's, kL eBAM's support a same common output pattern, where $k \in [0, 1]$. The condition for the output pattern of the network is the same as the one supported by the kL eBAM's is

$$k > \frac{1}{2} + \frac{1}{2 \cdot SNR_{eBAM}}. \tag{12}$$

If the lower bound in (12) is larger than one, then it means that even all of the eBAM's in the network support one output pattern, there is no guarantee to recall this output pattern.

By the above theorem, please note because the SNR_{eBAM} is usually very large, the lower bound of k can be simplified to be $\frac{1}{2}$ which complies the human intuition. That is, if more than 50% of the evidence supports a hypothesis, then the reasoning result most likely would be the same as this hypothesis.

E. Guaranteed Stable Condition of eBAM

By the meaning of evolution equations of eBAM, (3), the search of the desired pattern pairs is basically a back and forth reverberation process on the energy plane, (3). Certainly, it will take more time to correctly recall a pattern pair in this way. Hence, we are interested in discovering the conditions for recalling the desired pattern pair in "one shot." That is, we would like to discuss what the condition is to recall the correct pattern pair in one and only one back-and forth sweep. According to the first equation of (3)

$$\begin{aligned}
y_k &= sgn\left(\sum_{i=1}^{N} y_{ik} b^{X_i \cdot X_T}\right) \\
&= sgn\left(b^n y_{Tk} + \sum_{i \neq T}^{N} y_{ik} b^{X_i \cdot X_T}\right)
\end{aligned}$$

where X_T is one of the stored patterns in the eBAM. Thus, if we wish to recall the pair in one shot, then the pattern pair must be stored placing in a "stable state," i.e.,

$$| b^n y_{Tk} | \geq | \sum_{i \neq T}^{N} y_{ik} b^{X_i \cdot X_T} |$$
$$b^n \geq | \sum_{i \neq T}^{N} y_{ik} b^{X_i \cdot X_T} | .$$

Then, we have to know what the possible largest value of the right-hand side of the above equation is. The right-hand side of the above equation can be further derived to be

$$
\begin{aligned}
\left| \sum_{i \neq T}^{N} y_{ik} b^{X_i \cdot X_T} \right| &\leq \sum_{i \neq T}^{N} \left| y_{ik} b^{X_i \cdot X_T} \right| \\
&= \sum_{i \neq T}^{N} \left| y_{ik} \right| b^{X_i \cdot X_T} \\
&= \sum_{i \neq T}^{N} b^{X_i \cdot X_T} \\
&\leq \sum_{i \neq T}^{N} b^{n-2} \\
&= (N-1) b^{n-2}.
\end{aligned}
$$

Hence, we can state that the guaranteed stable state condition for the eBAM as follows.

Theorem of Guaranteed Stable State Condition of eBAM: Given an eBAM, the condition for one shot correct recall of any desired a pattern pair is

$$b^2 \geq N - 1 \tag{13}$$

where b is the base used in the eBAM evolution equations and N is the number of stored pattern pairs.

Note that the above theorem is a very strict theorem in terms of memory's capacity. It states a condition for one shot guaranteed recall of pattern pairs. Generally speaking, however, we do not demand the eBAM to recall the pattern pairs in one shot, because it will severely reduce the capacity. Hence, we would like to enlarge the capacity by relaxing the constraint, (13).

Lemma: Relaxation of guaranteed stable state condition by choosing a reasonable SNR, e.g., 10, is practically feasible for storing large amount of pattern pairs in eBAM.

$$b^4 \geq SNR \cdot 2^{-n+2}(N-1). \tag{14}$$

For example, if $b = e, SNR = 10, n = 16$, then we can store up to $N = 89454$ pattern pairs in a single eBAM.

III. Simulation Analysis

In this section, we use some examples to illustrate the theoretical results of the multi-eBAM networks discussed in Section II.

Example 1: This example is used to prove the bidirectional stability of multi-eBAM networks. As shown in Fig. 1, we set $L = 3, n = 7, p = 5, b = e$ in a multi-eBAM network. The training pattern pairs for the upper part of the network are as follows

$$
\begin{aligned}
A_{11} &= (1101101) \\
A_{12} &= (0000000) \\
A_{13} &= (1111111) \\
A_{14} &= (0100110) \\
A_{21} &= (1111111) \\
A_{22} &= (1000100) \\
A_{23} &= (1001101) \\
A_{24} &= (1100101) \\
A_{31} &= (0100011) \\
A_{32} &= (0100000) \\
A_{33} &= (1000001) \\
A_{34} &= (1001000) \\
B_1 &= (00101) \\
B_2 &= (11011) \\
B_3 &= (11000) \\
B_4 &= (00010)
\end{aligned}
$$

After the pattern pairs being transformed to the bipolar mode, three input pattern are presented at respective eBAM's

$$
\begin{aligned}
I_1 &= (\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ ) \\
I_2 &= A_{21} \\
I_3 &= A_{31}.
\end{aligned}
$$

The energy of the entire network according to (3) changes as follows: Iteration 0, E = -2.38782$\times 10^3$ Iteration 1, E = -2.82247$\times 10^3$ Iteration 2, E = -3.78538$\times 10^3$ Besides, the final pattern pairs of the network are

$$
\begin{aligned}
B_{final} &= B_1 \\
A_{1final} &= A_{11} \\
A_{2final} &= A_{21} \\
A_{3final} &= A_{31}.
\end{aligned}
$$

Note that the I_1 is not the same A_{11}, which is one bit away from I_1. Hence, not only does it show the stability of the network, but also it shows the error correcting ability. In the following, we will prove the majority rule by presenting different combination of input patterns. If the network is given

$$
\begin{aligned}
I_1 &= (\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ ) \\
I_2 &= A_{22} \\
I_3 &= A_{32}.
\end{aligned}
$$

The second and third eBAM's support the B_2 as the output pattern, but the first eBAM prefers B_1. The energy of the entire network according to (3) changes as follows: Iteration 0, E = -2.42522$\times 10^3$ Iteration 1, E = -2.86787$\times 10^3$ Iteration 2, E = -3.81306$\times 10^3$ The final pattern pairs of the network are

$$
\begin{aligned}
B_{final} &= B_2 \\
A_{1final} &= A_{12} \\
A_{2final} &= A_{22} \\
A_{3final} &= A_{32}.
\end{aligned}
$$

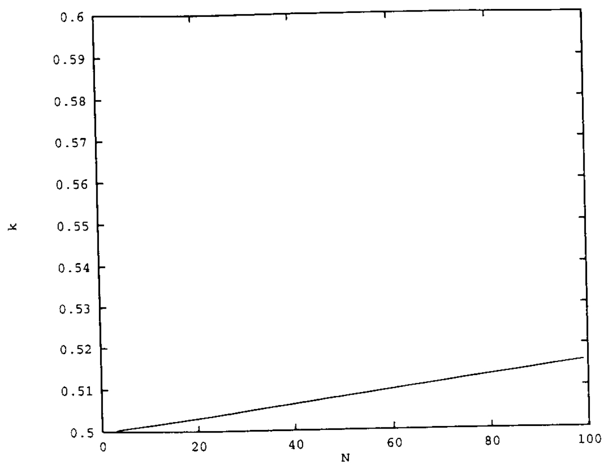

Fig. 2. The lower bound of k with different N.

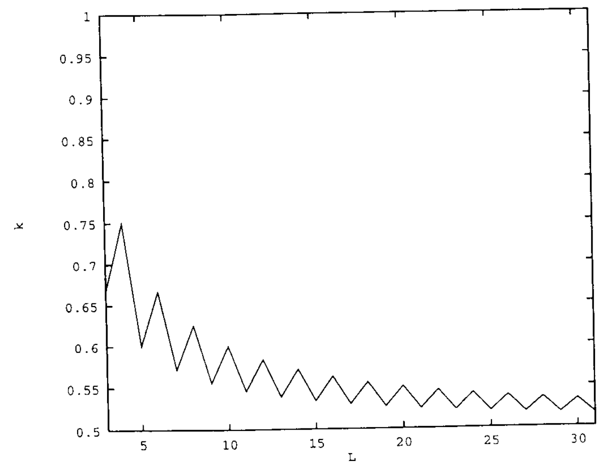

Fig. 3. The minimal k in different networks.

The result coincides with the prediction of majority rule, i.e., the final output pattern is B_2, because $k = \frac{2}{3}$ is much larger than the required minimal lower bound. Further more, if the network is given input patterns as

$$I_1 = (\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ )$$
$$I_2 = A_{23}$$
$$I_3 = A_{33}.$$

In this case, the second and third eBAM's support B_3 simultaneously. Then, the energy of the entire network according to (3) changes as follows: Iteration 0, E = -2.42789$\times 10^3$ Iteration 1, E = -2.87054$\times 10^3$ Iteration 2, E = -3.83341$\times 10^3$ Besides, the final pattern pairs of the network are

$$B_{final} = B_3$$
$$A_{1final} = A_{13}$$
$$A_{2final} = A_{23}$$
$$A_{3final} = A_{33}.$$

According to the above result, it is obvious when more than half of the eBAM's in the network agree to support a common output pattern, the output of the network will be this output pattern.

Example 2: A 3-eBAM network might not be considered to be a general case. Therefore, we construct a series of multi-eBAM networks, $L = 3$ to $L = 31$, $n = p = 8$, to verify the k prediction of the majority rule. In every multi-eBAM simulation, the number of stored pattern pairs for each single eBAM is also varied from $N = 3$ to $N = 99$. In this simulation, all of the pattern pairs are randomly generated. The result is the majority rule holds in every case, even in the worse case, $L = 31, N = 99$, in which the predicted $k = 0.515923$, and $16/31 = 0.516129 > 0.515923$. The relation between k and N is illustrated in Fig. 2, while the minimal k selected in networks with different number of eBAM's is shown in Fig. 3.

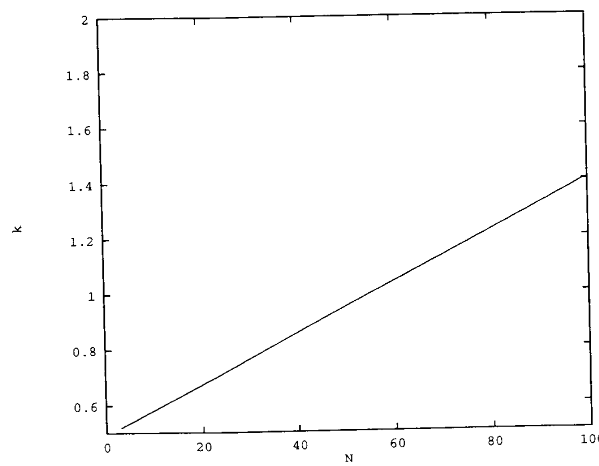

Fig. 4. The lower bound of k in the special case.

For the sake of comparison, we also repeat the simulation for the special case, which is also deemed as a strict case. The minimal k in this strict case is shown in Fig. 4 in which $b = e$. If the number of stored pattern pairs, N, is larger than 55, there is always a chance for the network to converge to the desired common output. Because when $N > 55$, the k is larger than one. According to (7), there is a chance for the network to recall the desired common output pattern when $N - 1 > b^4$. The comparison for the strict case and the general case is shown in Table I.

IV. Conclusion

A multi-eBAM neural network has been introduced for the belief combination in evidential reasoning. It is proved to be bidirectionally stable, which ensures the model's ability to reach a local energy minimum. The sufficient conditions for a multi-eBAM network guarantee the network to recover a specific, predetermined pattern pair from a list of choices.

TABLE I
THE MINIMAL k IN THE STRICT CASE AND THE GENERAL CASE. ($b = e$)

N (number of pairs)	strict case	general case
5	0.536631	0.500650
10	0.582420	0.501462
15	0.628209	0.502275
20	0.673999	0.503087
25	0.719788	0.503899
30	0.765577	0.504712
35	0.811366	0.505524
40	0.857155	0.506337
45	0.902944	0.507149
50	0.948733	0.507961
55	0.994522	0.508774
60	1.040311	0.509586
65	1.086100	0.510398
70	1.131890	0.511211
75	1.177679	0.512023
80	1.223468	0.512836
85	1.269257	0.513648
90	1.315046	0.514460
95	1.360835	0.515273

Most important of all is the theorem of the majority rule of the network proves this neural network complies with the intuition of human reasoning. Two majority rules and their respective bounds for the majority factor, k, are presented. These rules will help researchers to use and predict the result of evidential reasoning. The condition for the fastest recall in the eBAM network is also discovered, which states the bound of guaranteed stable states. This network provides the ability to process many evidence at the same time reaching a consented hypothesis.

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