Journal of Intelligent and Robotic Systems **10**: 147–171, 1994.

A Robust Continuous Model for Evidential Reasoning

CHUA-CHIN WANG
Department of Electrical Engineering, National Sun Yat-Sen University, Kaohsiung, Taiwan 80424, Republic of China

and

HON-SON DON*
Department of Electrical Engineering, National Chung-Hsing University, Taichung, Taiwan 40227, Republic of China

(Received: 14 May 1992; in final form: 12 May 1993)

Abstract. A 2-D model for evidential reasoning is proposed, in which the belief function of evidence is represented as a belief density function which can be in a continuous or discrete form. A vector form of mutual dependency relationship of the evidence is considered and a dependency propagation theorem is proved. This robust method can resolve the conflicts resulting from either the mutual dependency among evidences or the structural dependency in an inference network due to the evidence combination order. Belief conjunction, belief combination, belief propagation procedures, and AND/OR operations of an inference network based on the proposed 2-D model are all presented, followed by some examples demonstrating the advantages of this method over the conventional methods.

Key words. Reasoning, belief function, belief conjunction, mutual dependency, inference network, AND/OR graph, belief propagation

1. Introduction

Evidential reasoning, which has been an essential part of many computational systems, is the task of assessing a certain hypothesis when certain pieces of evidence are given. The hypothesis is assessed by inferring its *belief value* from the belief values of different pieces of evidence. The belief value can be taken from a certain belief region, e.g., a unit interval [0,1], which can be discrete or continuous, or from a set of linguistic quantifiers, e.g., [*very unlikely, unlikely, likely, very likely*], etc. Regardless of the type of representation, the belief value of an evidence indicates the *belief strength* of that evidence. In describing the relationship between different pieces of evidence, *dependency* has been employed to describe the degree of truth of one piece of evidence implied by a second piece of evidence. The belief strength of an evidence is not subject to any change of another evidence is a basic definition of *independency* of the evidence. On the other hand, if the belief

* All correspondences should be sent to the second author, Hon-Son Don.

strength of an evidence is subject to another piece of evidence, it is deemed that there is a *dependency* relationship between these two pieces of evidence. If a hypothesis is supported by many pieces of evidence, then the combined belief strength of the hypothesis is the belief value caused not only by the individual evidence but also the mutual dependencies among the pieces of evidence supporting the hypothesis.

Because of the obscure and inexact nature of information, each piece of evidence is associated with some *uncertainty*. Reasoning with uncertainties in an inference network [3] includes three types of uncertainty aggregations: belief conjunction, belief combination, and belief propagation. There are three major frameworks of evidential reasoning in the literature, i.e., the Dempster–Shafer theory of evidence, the fuzzy set theory, and the Bayesian probability theory [7, 12]. The advantages and disadvantages of these three frameworks have been discussed in [1, 2, 14, 15, and 16]. The ability of Shafer's belief function to manage uncertainty of information in a rule-based system has attracted much attention in artificial intelligence research. However, even with its strong popularity, the Shafer's model has drawbacks. Shafer's belief function model uses numerical values in the interval [0,1] to represent the degree of belief of information, which can also be interpreted as an index of inexactness of that information. The nonrobustness of this model has been discussed in [5, 17]. Secondly, the basic probability assignment (BPA) of a belief function is in a form of discrete type function which can not always provide a precise description of an evidence for all the situations. It is often not appropriate to assign a discrete basic probability assignment over [0,1] by thresholding the interval into several regions, since the thresholds themselves can not describe the exact nature of an evidence. The possible quantization problem caused by thresholding a continuous region for the weight of evidence has been discussed in [1], which provides an example showing that a quantized version of an associative belief combination may not necessarily be an associative one. The continuous form of a belief function, which is a more general representation, is needed to approximately express the vagueness of an evidence. An example which can be properly managed by neither Shafer's model nor the fuzzy set theory will be given in Section 2.1. Several other belief function approaches are included in [4] and [8], which have focused on handling the belief combination problem by using Dempster's rule to deal with belief propagation. However, these approaches didn't provide a formal proof of their respective methods. Other previous work includes Shafer and Logan's algorithm for hierarchically structured hypotheses [11], and an improved algorithm by Shafer and Shenoy [13]. In a rule-based intelligent system, the inconsistency of evidential reasoning resulting from the mutual dependency among different pieces of evidence and the structural dependency caused by the improper arrangement of an inference network has not been fully solved. One example to illustrate this inconsistency will be provided in Section 2.3.

Although much effort has been spent on belief combination, the uncertainty management of continuous belief function and the conflict due to the dependency of evidence are still not solved. The following is a proposed method which focuses on achieving conflict resolution of belief combination resulting from mutual dependency of evidence in an inference network. In addition, this method can also handle the information aggregation based upon the continuous belief functions, which is closer to the human reasoning process. The nature of a belief function associated with an evidence is a probabilistic function, which could be discrete or continuous. We will discuss the belief conjunction first, since the belief combination and the belief propagation are both based on the belief conjunction. The belief combination is the belief conjunction of many pieces of evidence supporting the same hypothesis, while the belief propagation is the belief conjunction of a piece of evidence and a rule [5].

This paper is organized as follows: Section 2 presents a new representation of the uncertainty of an evidence and an inference rule. The theory of the proposed model, called *2-D model*, is discussed and applied to establish a procedure of handling belief conjunction, belief combination, belief propagation, and AND/OR operations. Section 3 provides an example showing the advantages of the 2-D model, followed by the conclusion in Section 4.

2. Theory of the 2-D Model

2.1. REPRESENTATION OF EVIDENCE AND RULE

The first step in the simulation of human reasoning with uncertainty is to find a proper way to represent the uncertainty and then build up the inference procedure. In the belief function introduced by Shafer [12] and Hau [5], two parameters, i.e., lower bound and upper bound, are employed to indicate *credibility* and *plausibility*. For the sake of clarity, the *belief function* is borrowed to represent the belief density function associated with an evidence in the following text, and the belief function proposed by Shafer, [11–13] is named as *Shafer's belief function*. But, as discussed earlier, the probability assignment strategy for Shafer's belief function has its inherent drawback. For example, if a piece of evidence is to emphasize that the closer it is to the truth, the stronger it is, then that evidence can be conveniently modeled by a linear continuous function, which is a density function,

$$\mathrm{Bel}(\theta) = k \cdot \theta, \tag{1}$$

where θ is in the interval [0,1] indicating the authenticity of the evidence, and k is a constant. We can hardly find any significant thresholds to quantize the associated belief function into a discrete form which can be handled by either Dempster–Shafer theory [2] or Hau's modified Dempster's rule [5]. Therefore, a more general representation of evidence is needed to represent such kind of uncertainty. In the following, we present our representation of the evidence and the inference rules.

DEFINITION 1. A piece of evidence in a rule-based system is represented by a subset A of the frame of discernment Θ, and a belief function associated with A is represented by a belief density function $p_A(\theta)$, where θ is a variable indicating the degree of truth for the evidence. $\bar{A}$ denotes the complement of A. 1 is used to denote the truth of the evidence and 0 is used to denote the falsity of the evidence. θ is a numerical value in the interval [0,1]. The total amount of belief in the interval [0,1] is

$$\int_0^1 p_A(\theta)\,\mathrm{d}\theta = 1. \tag{2}$$

In order to avoid any confusion, this representation is called the *belief density function* which is a function to describe the distribution of a fixed amount of belief, say 1, in an interval [0,1]. This type of belief density function can be transformed into a Shafer's belief function by assigning two bounds to the interval. For instance, if the above linear continuous belief function, equation (1), is going to be transformed into the conventional belief function by choosing two thresholds in the belief region [0,1] and compute the respective area of each part so that the numerical values of the credibility and the plausibility of this belief function are obtained. If two thresholds, say 1/3 and 2/3, are chosen and k is 2 derived from equation (2), the following results are obtained,

$$\mathrm{Cr} = \int_{2/3}^1 2\theta\,\mathrm{d}\theta = \frac{5}{9}, \qquad \mathrm{Pl} = \int_{1/3}^1 2\theta\,\mathrm{d}\theta = \frac{8}{9}.$$

On the other hand, given a Shafer's belief function by BPA method, e.g., a belief function with credibility 5/9 and plausibility 8/9, it can not precisely express the characteristics of the linear continuous belief function shown by equation (1).

DEFINITION 2. A rule R in a rule-based system conveying uncertainty is represented as

$$R\colon\ E \longrightarrow H \quad \text{with} \quad p_{E\to H}(\theta),$$

where E is an evidence, H is a hypothesis implied by E, and $p_{E\to H}(\theta)$ is a belief density function to describe the degree of the truth of the rule. E is called *antecedent*, and H is called *consequent* of rule R.

In the above definition, the rule R: $E \to H$ is interpreted as logic implication. In Section 1, we mentioned the inconsistency of evidential reasoning resulting from the mutual dependency among different pieces of evidence and the structural dependency caused by the various arrangements of an inference network. One example showing such inconsistency will be given in Section 2.3. The following definition of degree of *mutual dependency* is given to describe the relationship between two pieces of evidence.

DEFINITION 3. The degree of an evidence A depending on another evidence B is represented by ρ_{AB}, where by $0 \leqslant \rho_{AB} \leqslant 1$.

Note that ρ_{AB} is not necessarily equal to ρ_{BA}, which means a dependency relationship is a directed link. If ρ_{AB} equals 1, then it indicates A is totally dependent on B; if ρ_{AB} is 0, then A won't be affected by B at all. In the following, $\rho_{A\Rightarrow B}$ is used to denote the belief function of evidence A depending on evidence B, and ρ_{AB} is used to denote the coefficient of A depending on B.

2.2. BELIEF CONJUNCTION

By definition, belief conjunction refers to the deduction of the belief associated with $(A \cap B)$ from the belief associated with evidence A and B, respectively. That is, given two frames of discernment Θ_A and Θ_B, a compatibility relation between Θ_A and Θ_B is the Cartesian product of them, which is represented as

$$\Theta_A \times \Theta_B \longrightarrow \Theta_{A\cap B}. \tag{3}$$

There are three possible dependency relationships between two pieces of evidence A and B, which are

(i) A and B are independent;
(ii) A depends on B; and
(iii) B depends on A.

The relationship between two pieces of evidence can not be explicitly expressed by only one of the above relationships due to the vagueness and incompleteness of the evidence. In the following, we will develop a general formulation to cope with such a situation. Let

$$\mathrm{Conj}(A, B) = A \cap B$$

and

$$\begin{aligned} p_{A\cap B}(\theta) &= \rho_I \cdot p_{\mathrm{indep}}(\theta) + \rho_{AB} \cdot p_{A\Rightarrow B}(\theta) + \rho_{BA} \cdot p_{B\Rightarrow A}(\theta) \\ &= [\rho_I \, \rho_{AB} \, \rho_{BA}] \cdot \begin{bmatrix} p_{\mathrm{indep}}(\theta) \\ p_{A\Rightarrow B}(\theta) \\ p_{B\Rightarrow A}(\theta) \end{bmatrix} \\ &= \alpha \cdot \mathbf{P}, \end{aligned} \tag{4}$$

where

$$\alpha = [\rho_I \, \rho_{AB} \, \rho_{BA}],$$

$$\mathbf{P} = \begin{bmatrix} p_{\text{indep}}(\theta) \\ p_{A \Rightarrow B}(\theta) \\ p_{B \Rightarrow A}(\theta) \end{bmatrix}.$$

The ρ's are dependency coefficients, which represent the degree of dependencies and $\rho_I + \rho_{AB} + \rho_{BA} = 1$. The α is called *dependency vector*, and the $\mathbf{P}$ is called the *belief function vector.* There are three terms in (2), representing three different types of belief conjunction. Their meanings and how they can be computed are discussed in the following.

Case 1: Independent

Referring to Figure 1, since the two pieces of evidence A and B are independent, we assume that they can be located, respectively, on the two axes μ and ν of the Cartesian coordinates. The belief density function of the $A \times B$ is the multiplication of the probability density functions associated with individual evidence. That is,

$$p_{\text{indep}}(\mu, \nu) = p_A(\mu) \cdot p_B(\nu). \tag{5}$$

The above expression is a 2-D function forming a 2-D surface which should be reduced to 1-D form for the belief function of the conjunction result. Since these two pieces of evidence are independent with each other, they are considered to be equally important to the desired belief conjunction, $\text{Conj}_{\text{indep}}$, i.e., their contribution to the final result is the same. Therefore, for a ω, all of the density products $p_A(\mu) \cdot p_B(\nu)$, where $\mu + \nu = \omega$, should be attributed to $p_{\text{indep}}(\omega)$. The line $\mu = \nu$ is called *conjunction line*, and the resulted $p_{\text{indep}}(\omega)$ can be considered to reside on this line by the

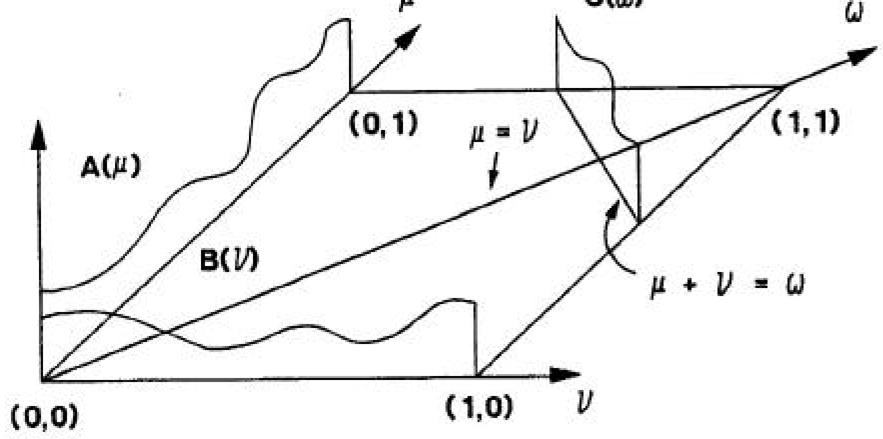

Fig. 1. The independent case of the 2-D model.

following calculations,

$$p_{\text{indep}}(\omega) = p_{\text{indep}}(\mu,\nu)|_{\omega=\mu+\nu}$$
$$= \begin{cases} \int_0^{\omega} p_A(\mu)\cdot p_B(\omega-\mu)\,\mathrm{d}\mu = \int_0^{\omega} p_B(\nu)\cdot p_A(\omega-\nu)\,\mathrm{d}\nu, \\ \qquad\qquad\qquad\qquad \text{if } 0 \leqslant \omega \leqslant 1, \\ \int_{\omega-1}^{1} p_A(\mu)\cdot p_B(\omega-\mu)\,\mathrm{d}\mu = \int_{\omega-1}^{1} p_B(\nu)\cdot p_A(\omega-\nu)\,\mathrm{d}\nu, \\ \qquad\qquad\qquad\qquad \text{if } 1 \leqslant \omega \leqslant 2, \end{cases} \tag{6}$$

where $\omega = \mu + \nu$. Obviously, the range of ω is [0,2]. Also note that $\mu + \nu = \omega$ is a line perpendicular to the line $\mu = \nu$. The following probability conservation property can be easily proved.

LEMMA 1.

$$\int_0^2 p_{\text{indep}}(\omega)\,\mathrm{d}\omega = 1. \tag{7}$$

Proof.

$$\begin{aligned} &\int_0^2 p_{\text{indep}}(\omega)\,\mathrm{d}\omega \\ &= \int_0^1\int_0^{\omega} p_A(\mu)\cdot p_B(\omega-\mu)\,\mathrm{d}\mu\,\mathrm{d}\omega + \int_1^2\int_{\omega-1}^{1} p_A(\mu)\cdot p_B(\omega-\mu)\,\mathrm{d}\mu\,\mathrm{d}\omega \\ &= \int_0^1\int_0^{1-\mu} p_A(\mu)p_B(\nu)\,\mathrm{d}\nu\,\mathrm{d}\mu + \int_0^1\int_{1-\mu}^{1} p_A(\mu)\cdot p_B(\nu)\,\mathrm{d}\nu\,\mathrm{d}\mu \\ &= \int_0^1\int_0^1 p_A(\mu)p_B(\nu)\,\mathrm{d}\nu\,\mathrm{d}\mu \\ &= 1. \end{aligned}$$

Lemma 1 is to prove the conservative property of total belief amount in an independent case of belief conjunction. However, we are aware of the assumed range of a belief function is [0,1]. Therefore, normalization of the conjuncted belief function is necessary. This can be done by the following equation.

$$p_{\text{indep}}(\theta) = 2p_{\text{indep}}(\omega)|_{\omega=2\theta}. \tag{8}$$

Case 2: Totally Dependent

This case includes two cases which are either A is totally dependent on B or B on A. Here we only discuss the former case, because the latter is the same. Referring to Figure 1, if A is totally dependent on B, then the resulting conjunction belief function should be the same as the belief function of B. This indicates that the $\mathrm{Conj}_{A\Rightarrow B}(A, B)$ is exactly the B, that is, the conjunction line is rotated to overlap the ν-axis and the range of ω, which is the variable of the $\mathrm{Conj}_{A\Rightarrow B}(A, B)$, is [0,1]. On the other hand, in the case of B depending on A, we will have the conjunction line overlapping the μ-axis, and the $\mathrm{Conj}_{B\Rightarrow A}(A, B)$ is exactly the A. Therefore, we have the following

$$p_{A\Rightarrow B}(\theta) = p_B(\theta), \tag{9}$$

$$p_{B\Rightarrow A}(\theta) = p_A(\theta). \tag{10}$$

The above cases 1 and 2 represent the extreme cases. The general case will lie in between these extreme cases and can be computed as an interpolation of these extreme cases according to the mutual dependency coefficients ρ_I, ρ_{AB} and ρ_{BA}. The result has been given in equation (4). The following conservation result can be easily proved.

LEMMA 2.

$$\int_0^1 p_{\mathrm{Conj}(A,B)}(\theta)\,\mathrm{d}\theta = 1.$$

Proof.

$$\begin{aligned}\int_0^1 p_{\mathrm{Conj}(A,B)}(\theta)\,\mathrm{d}\theta &= \rho_I \int_0^1 p_{\mathrm{indep}}(\theta)\,\mathrm{d}\theta + \\ &\quad + \rho_{AB}\int_0^1 p_{A\Rightarrow B}(\theta)\,\mathrm{d}\theta + \rho_{BA}\int_0^1 p_{B\Rightarrow A}(\theta)\,\mathrm{d}\theta \\ &= \rho_I \cdot 1 + \rho_{AB}\cdot 1 + \rho_{BA}\cdot 1 \\ &= 1.\end{aligned}$$

Lemma 2 is to prove the conservative property of belief amount in a belief conjunction procedure containing three cases, totally independent, A depending on B, and B depending on A. Lemma 1 is a special case of Lemma 2 when $\rho_{AB} = \rho_{BA} = 0$.

2.3. BELIEF COMBINATION

Belief combination refers to the belief conjunction of several pieces of evidence supporting the same goal hypothesis. Hau discussed in [5] a fact that the basic

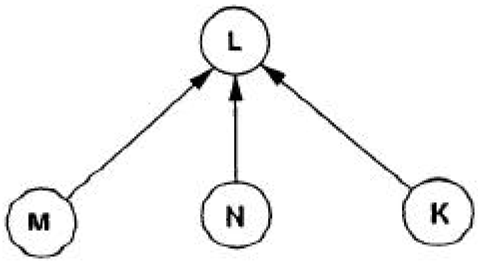

Fig. 2. Combination of three pieces of evidence to support a hypothesis.

probability assignment method will introduce a conflict ($A \cap \bar{A}$) in belief combination because of the conjunction of evidence supporting the same hypothesis with different degree of belief. Therefore, a conflict resolution approach is needed to achieve the consistency of belief combination. By the theory of previous subsection, if both A and B are supporting hypothesis C, then the belief density function shown on the conjunction line is the belief function of C. Therefore, equations (4) to (10) can also be applied to the combination of the two pieces of evidence, except that the two pieces of evidence must support the same hypothesis. The deficiency and nonrobustness of Dempster's rule have been detailedly discussed in Hau's work [5]. However, both Hau and Shafer ignored the vagueness of the dependency relationship between the two pieces of evidence. The mutual dependency relationship of pieces of evidence always introduces significant conflict in the reasoning result of an inference network. This conflict is shown in Figures 2 and 3. Referring to Figure 2, if there are three pieces of evidence, M, N, and K, supporting a hypothesis L, then in a sequential rule-based system, two of them have to be combined first, and then the result is combined with the third evidence. These two cases are shown in Figure 3.

EXAMPLE 1. Considering the two cases in Figure 3, which have different structures. Assume

$$\mathrm{Cr}(M) = 0.98, \quad \mathrm{Pl}(M) = 0.99, \quad \rho_{NK} = \rho_{KN} = 0.5,$$

$$\mathrm{Cr}(N) = 0.01, \quad \mathrm{Pl}(N) = 0.02, \quad \rho_{NM} = 0.1,$$

$$\mathrm{Cr}(K) = 0.01, \quad \mathrm{Pl}(K) = 0.99, \quad \rho_{KM} = 0.9.$$

The inconsistent results of these two cases by Hau's method are tabulated in Table I.

When compared with human's reasoning process, this kind of inconsistency is unimaginable. From the assumption, evidence M is the strongest one to support the hypothesis L, the other two pieces of evidence N and K are less important than M. According to Table I, the resulting credibility of case 1 is more than twice of that of case 2. On the contrary, the plausibility of case 1 is only half of that of case 2.

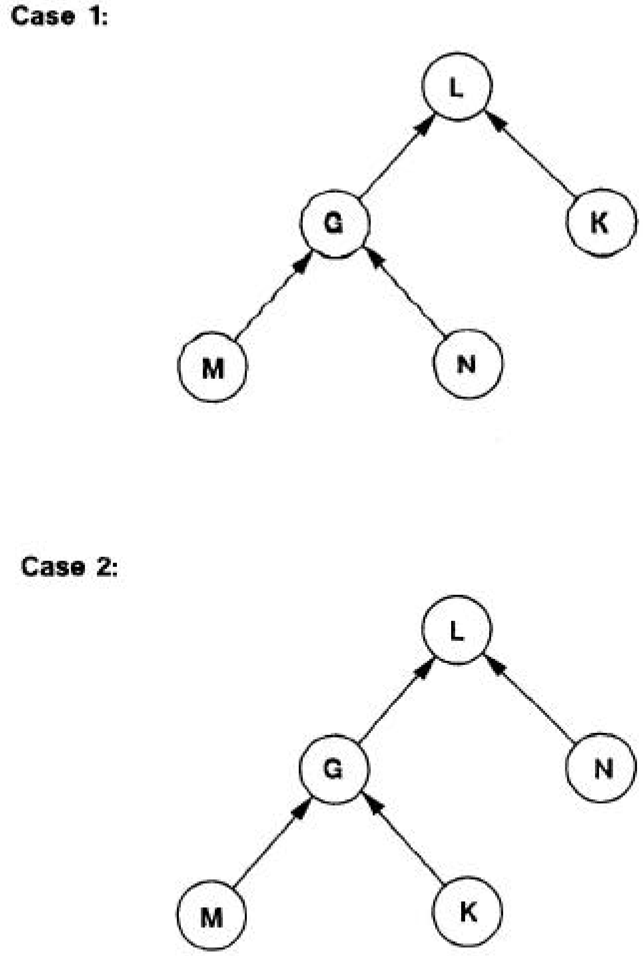

Fig. 3. Two different structures of combination of three pieces of evidence.

Table I. The results of Hau's approach applied to cases of Figure 3.

	Cr_L	Θ_L	$1 - \mathrm{Pl}_L$
Case 1	0.006903	0.012857	0.980239
Case 2	0.003064	0.033357	0.963579

These results indicate that Hau's method is easily subject to the combination order of the evidence, which is not consistent with the intuition of human reasoning. To a human, if these three pieces of evidence are given, a belief function associated with L should be dominated by M, since the relative dependency ratios of N and K indicate their less influence on L, and because K has a stronger dependency on M

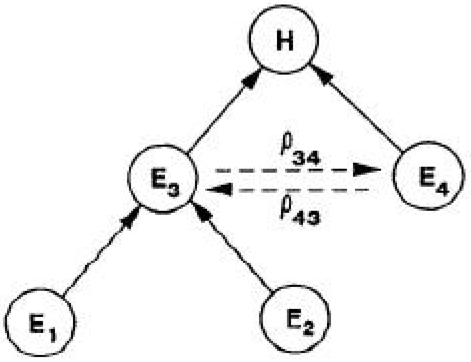

Fig. 4. An inference network for demonstrating the mutual dependency relationship between E_3 and E_4.

than N, M should have the dominant influence on L. Therefore, we conclude that the results given by Hau's method are not consistent with human reasoning.

The reason why the conflict appears in Example 1 is that the mutual dependency relationship will propagate through the inference network by current belief combination and then influence the following belief combination. Therefore, let's consider the situation that the pieces of evidence are arranged in a lattice-structured inference network, as shown in Figure 4, two pieces of evidence E_1 and E_2 are combined so as to form an *intermediate evidence* E_3, and then E_3 and E_4 are combined to support the hypothesis H. *Intermediate evidence* means an evidence synthesized by the other evidence, and has all the properties that a real evidence has. What have been given are the mutual relationships among E_1, E_2, and E_4. Therefore, before E_3 and E_4 are combined, the relationship between E_3 and E_4 has to be determined. The following information is assumed to be known before the belief combination is performed.

$$\rho_{12} + \rho_{21} + \rho_{I_{12}} = 1, \qquad \rho_{14} + \rho_{41} + \rho_{I_{14}} = 1, \qquad \rho_{24} + \rho_{42} + \rho_{I_{42}} = 1,$$

where all ρ's are known, and ρ_I's denote the dependency coefficients. The following result can be proved.

DEPENDENCY PROPAGATION PROPOSITION. *For the inference network shown in Figure 4 and the above given information, the mutual relationships of E_3 and E_4 can be determined by the following equations*

$$\begin{aligned}
\rho_{34} &= \frac{\rho_{12}}{\rho_{12}+\rho_{21}} \cdot \rho_{24} + \frac{\rho_{21}}{\rho_{12}+\rho_{21}} \cdot \rho_{14}, \\
\rho_{43} &= \frac{\rho_{12}}{\rho_{12}+\rho_{21}} \cdot \rho_{42} + \frac{\rho_{21}}{\rho_{12}+\rho_{21}} \cdot \rho_{41}, \\
\rho_{I_{34}} &= \frac{\rho_{12}}{\rho_{12}+\rho_{21}} \cdot \rho_{I_{42}} + \frac{\rho_{21}}{\rho_{12}+\rho_{21}} \cdot \rho_{I_{14}}.
\end{aligned} \tag{11}$$

Proof. Using the vector notation of Section 2.2, let the dependency vectors α_{14}, α_{24}, α_{34} denote the mutual dependency relationships between E_1 and E_4, E_2 and E_4, E_3 and E_4, respectively,

$$\alpha_{14} = [\rho_{I_{14}}\ \rho_{14}\ \rho_{41}],$$

$$\alpha_{24} = [\rho_{I_{24}}\ \rho_{24}\ \rho_{42}],$$

$$\alpha_{34} = [\rho_{I_{34}}\ \rho_{34}\ \rho_{43}].$$

Since E_3 is the combination result of E_1 and E_2, the following equation holds,

$$\alpha_{34} = \frac{\rho_{12}}{\rho_{12} + \rho_{21}} \cdot \alpha_{24} + \frac{\rho_{21}}{\rho_{12} + \rho_{21}} \cdot \alpha_{14}$$

which proves the proposition.

EXAMPLE 2. Given the same data as in Example 1, we use our method, i.e., equation (4) to (11), to process each combination in the two cases in Figure 3. The final results are listed in Table II.

In Table II, it is obvious that both cases have almost the same credibility Cr_L and plausibility $\mathrm{Cr}_L + \Theta_L$, which means our model will not be seriously influenced by the order of the belief combination compared with the result derived by Hau's modified Dempster's rule, and meets the intuition of human reasoning. The conflict appearing in Example 1 has been resolved by our model. Here we have introduced a vector form for mutual dependency shown in (4) and the above dependency propagation proposition, which turns out to be superior to the conventional scalar form for the dependency between different pieces of evidence in resolving conflict caused by the dependency problem.

Table II. The results of proposed model applied to either case of Figure 3.

	Cr_L	Θ_L	$1 - \mathrm{Pl}_L$
Case 1	0.106148	0.879853	0.013998
Case 2	0.105937	0.868451	0.025612

2.4. BELIEF PROPAGATION

Belief propagation refers to the aggregation of the uncertainty associated with the evidence or fact to fire a rule and the uncertainty of the rule itself so as to deduce the uncertainty of the goal hypothesis of the rule. Referring to Figure 5, given a rule R: $P \rightarrow Q$ and an evidence P, then we are interested in exploring the belief function of the conclusion Q supported by the evidence P, which is defined as the belief propagation result of the evidence P and the rule R. Because it is impossible

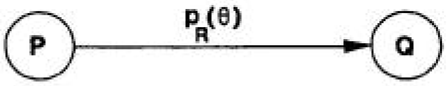

Fig. 5. Representation of an inference rule R.

to obtain the exact belief function of the consequent Q from the given evidence and rule, what we can expect is an assessment of the *maximum bound* and *minimum bound* of the belief function associated with Q. If an evidence A is covered by another evidence B, i.e., the information of A is contained in that of B, then we denote their relationship by $A \subseteq B$. Using this notation, the relationship among the maximum bound $Q_{\max}$, the minimum bound $Q_{\min}$, and Q can be espressed as

$$Q_{\min} \subseteq Q \subseteq Q_{\max}.$$

Let $p_{P \to Q}(\theta)$ be the belief function associated with the rule R and $p_P(\theta)$ be the belief function associated with the evidence P. The conjunction of the rule and the evidence is

$$\begin{aligned}(P \longrightarrow Q) \cap P &= (\bar{P} \cup Q) \cap P \\ &= (\bar{P} \cap P) \cup (Q \cap P) \\ &= (Q \cap P).\end{aligned}$$

The conjunction result $(Q \cap P)$ means the consequent Q holds when it is supported by the antecedent P. Therefore, the belief function of this conjunction provides the minimum bound of the belief function of Q, i.e., $Q_{\min} = (Q \cap P)$. This result meets the definition which we have explored for the belief propagation. Therefore, by applying the conjunction procedure of Section 2.2 to an evidence and a rule, we will get the result of belief propagation. In other words, the belief function obtained by the conjunction of the belief functions of the antecedent and the rule is actually the minimum bound of the belief function of the consequent.

However, we are also interested in assessing the maximum bound of the belief function of the consequent Q. Referring to [4] and [5], the smallest range of $\bar{Q}$ is $(P \cap \bar{Q})$ which can also be derived from basic logic operations. Hence, we conclude that

$$\begin{aligned}Q_{\max} &= \overline{(P \cap \bar{Q})} \\ &= P \longrightarrow Q,\end{aligned}$$

which implies the maximum bound of the belief function of Q is exactly the same as the belief function of the given rule despite what the belief function of P is. In other words, if the maximum bound is employed as the belief propagation result, then the uncertainty of the antecedent will not have any influence on the belief propagation. This is not true at all. Note that many researchers have proposed various approaches to recover and assess the belief function of the consequent, and provided many explanations to their methods, e.g., [4] and [5]. However, we are only interested in the influence provided by the antecedent to the consequent which shows the degree of the antecedent supporting the consequent. Henceforth, we adopt only the minimum bound of the belief function of the consequent in a belief propagation procedure to assess the uncertainty aggregation of belief propagation.

In propositional logic, the logic implication, $A \rightarrow C$, can be synthesized by the conjunction of other logic implications, for example, the rules $A \rightarrow B$ and $B \rightarrow C$. It is easy to derive the following '*chaining syllogism*'. Given two rules

$$R_1\text{: } A \longrightarrow B$$

$$R_2\text{: } B \longrightarrow C,$$

with the belief functions $p_{R_1}(\theta)$ and $p_{R_2}(\theta)$, respectively. The belief function $p_{R_3}(\theta)$ associated with the new rule, R_3: $A \rightarrow C$, is the conjunction of the two belief functions, $p_{R_1}(\theta)$ and $p_{R_2}(\theta)$.

2.5. AND/OR OPERATIONS

In an inference network, the function of each node is either AND or OR operation. In order to analyze the uncertainty aggregation of an inference network, it is necessary to consider these two operations for belief function. In Section 2.2, we mentioned that the conjunction of two pieces of evidence are deemed as the AND operation of the two pieces of evidence, which is described as $\mathrm{Conj}(A, B) = A \cap B$. Therefore, all of the theory of the belief conjunction given in Section 2.2 can be applied to the AND operation.

An OR operation for two pieces of evidence can be defined as $\mathrm{Union}(A, B) = A \cup B$. From basic logic theory, we know

$$A \cup B = A + B - A \cap B,$$

which means the following equation holds,

$$p_{\mathrm{Union}}(\theta) = p_A(\theta) + p_B(\theta) - p_{\mathrm{Conj}}(\theta) \tag{12}$$

where $p_{\mathrm{Conj}}(\theta)$ can be derived according to the model introduced in Section 2.2.

Since an inference network can be viewed as an AND/OR graph, if all of the evidence, rules, and the mutual dependency relationships are given, the belief function of the hypothesis can be derived by the proposed model. The procedure is first to perform the operations of low-level nodes and then gradually propagate towards the root where the hypothesis resides. For a given inference network structure, we can arbitrarily assign the AND/OR operation to each node. In the following, we use the inference network structure of case 1 in Figure 3 to illustrate the computation of the belief functions of the hypothesis under various configurations.

1. AND Network: Suppose we are given the following rules and mutual dependency relationships:

$$R_1\text{: if } (M \cap N) \longrightarrow G, \quad p_{R_1}(\theta)$$

$$R_2\text{: if } (G \cap K) \longrightarrow L, \quad p_{R_2}(\theta)$$

$$p_M(\theta), \quad p_N(\theta), \quad p_K(\theta),$$

and

$$\rho_{xy}, \quad \text{where } x, y = M, N, K, R_1, R_2.$$

The above information shows that all of the nonterminal nodes are AND nodes. In order to assess the uncertainty of the hypothesis, we apply the following procedure: (1) Compute the belief conjunction of evidence M and N; (2) Apply the rule R_1 to obtain the belief function of the intermediate evidence G; (3) Compute the conjunction of G and K in which the dependency propagation must be taken into account; (4) Apply the rule R_2 to obtain the belief function of the hypothesis L.

2. OR Network: If we are given the following rules, which are different from the previous case, and the mutual dependency relationships:

$$R_1\text{: if } M \longrightarrow G, \quad p_{R_1}(\theta)$$

$$R_2\text{: if } N \longrightarrow G, \quad p_{R_2}(\theta)$$

$$R_3\text{: if } G \longrightarrow L, \quad p_{R_3}(\theta)$$

$$R_4\text{: if } K \longrightarrow L, \quad p_{R_4}(\theta)$$

$$p_M(\theta), \quad p_N(\theta), \quad p_K(\theta),$$

and

$$\rho_{xy}, \quad \text{where } x, y = M, N, K, R_1, R_2, R_3, R_4.$$

The above information shows that all of the nonterminal nodes are OR nodes. To assess the belief function of the hypothesis L, we follow the following procedure: (1) Apply rules R_1 and R_2 to evidence M and N, respectively, to obtain two belief functions for the intermediate evidence G; (2) Use equation (10) to compute the 'union' of these two belief functions to produce the belief function of G; (3) Apply rules R_3 and R_4 to evidence G and K, respectively, to obtain two belief functions for the intermediate node L; (4) Compute the belief function of the hypothesis L by the 'union' operation in equation (10).

3. Mixed AND/OR Network: It is also possible that both the AND and OR nodes exist in an inference network. Assume the following information is given:

R_1: if $M \longrightarrow G$, $p_{R_1}(\theta)$

R_2: if $N \longrightarrow G$, $p_{R_2}(\theta)$

R_3: if $(G \cap K) \longrightarrow L$, $p_{R_3}(\theta)$

$p_M(\theta)$, $p_N(\theta)$, $p_K(\theta)$,

and

ρ_{xy}, where $x, y = M, N, K, R_1, R_2, R_3$.

The above information shows that the node G is an OR node and the node L is an AND node. We can employ the following procedure to assess the uncertainty of the hypothesis: (1) Apply rules R_1 and R_2 to evidence M and N, respectively, to obtain two belief functions for the intermediate evidence G; (2) Compute the 'union' of the these two belief functions to produce the belief function of G; (3) Compute the conjunction of G and K; (4) Apply rule R_3 to the previous result and obtain the belief function of the hypothesis L.

Note that for all belief conjunction, union, and propagation steps, the dependency propagation must be taken into consideration so as to avoid any possible conflict.

3. Simulation Examples

In this section, we use some examples to illustrate the theory and procedures of the proposed evidential reasoning model presented in the previous sections.

EXAMPLE 3. There are two pieces of evidence A and B supporting a hypothesis C. Assume the property of A supporting C is the more A is true, then the more C is also true. However, if B has an opposite property to that of A, and B is partially dependent upon A with an degree 0.4, then what is the belief that A and B support C?

We know that the properties of evidence A and B given in the above can not be easily modeled by the *basic probability assignment* (BPA) method to be processed by the Shafer–Dempster's rule. However, it can be handled by the proposed model. First, we can assign a belief function to the evidence A based on its property,

$$p_A(\theta) = 2\theta.$$

On the other hand, we can assign a belief function to the evidence B,

$$p_B(\theta) = 2 - 2\theta.$$

Following the procedure discussed in Sections 2.2 and 2.3, first, three dependency coefficients must be computed. Since there is no information to indicate there is a possibility that A depends upon B either partially or totally, we can assume that

$$\rho_{B\Rightarrow A} = 0.4, \qquad \rho_{A\Rightarrow B} = 0.0, \qquad \rho_{I_{AB}} = 0.6.$$

Secondly, we calculate the $p_{\text{indep}}(\theta)$, which can be deduced by equations (3), (4), (5), and (6).

$$p_{\text{indep}}(\theta) = \begin{cases} 16\theta^2 - \frac{32}{3}\theta^3, & 0 \leqslant \theta \leqslant \frac{1}{2}, \\ \frac{16}{3} - 16\theta^2 + \frac{32}{3}\theta^3, & \frac{1}{2} \leqslant \theta \leqslant 1. \end{cases}$$

Then, since A and B support the same hypothesis, we can apply (2) to get the overall belief function of hypothesis C, which is

$$p_C(\theta) = 0.4p_A(\theta) + 0.6p_{\text{indep}}(\theta) = \begin{cases} \frac{4}{5}\theta + \frac{48}{5}\theta^2 - \frac{32}{5}\theta^3, & 0 \leqslant \theta \leqslant \frac{1}{2}, \\ \frac{16}{5} + \frac{4}{5}\theta - \frac{48}{5}\theta^2 + \frac{32}{5}\theta^3, & \frac{1}{2} \leqslant \theta \leqslant 1. \end{cases}$$

The graphs of belief functions, $p_A(\theta)$, $p_B(\theta)$, and $p_{\text{indep}}(\theta)$ are shown, respectively, in Figures 6, 7, and 8. The graph of the belief function $p_C(\theta)$ associated with the

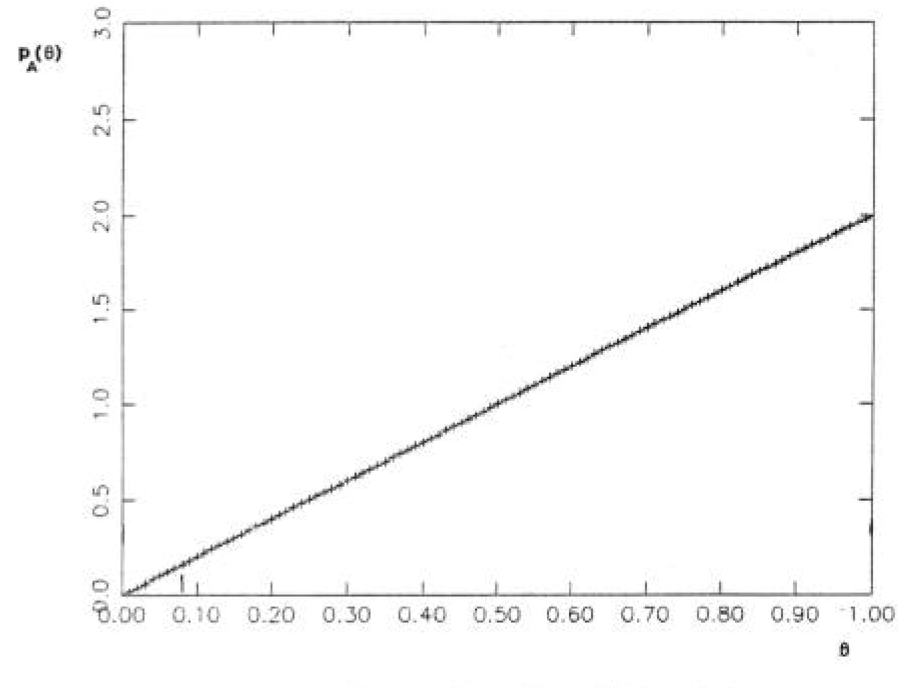

Fig. 6. Belief function of the evidence A in Example 3.

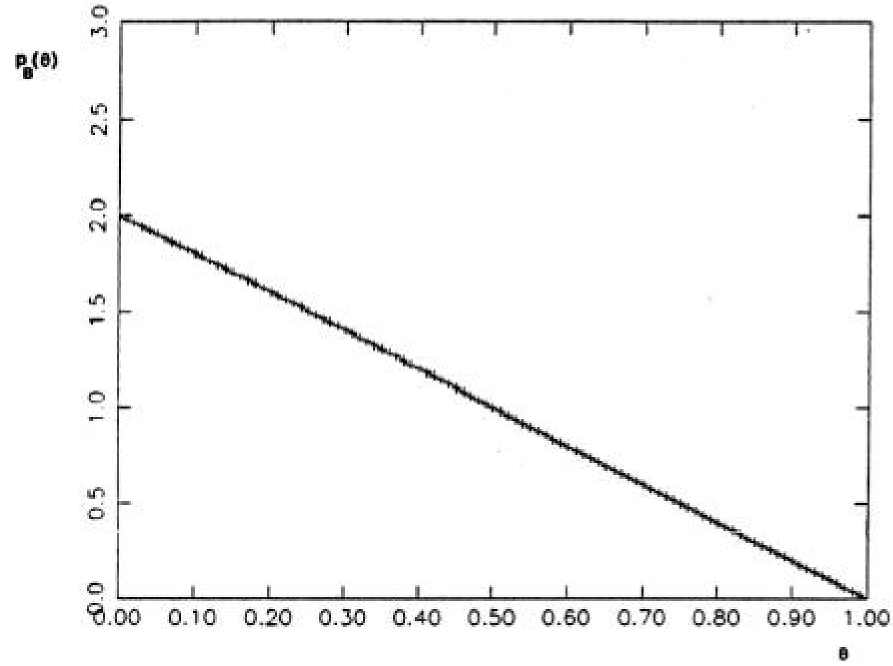

Fig. 7. Belief function of the evidence B in Example 3.

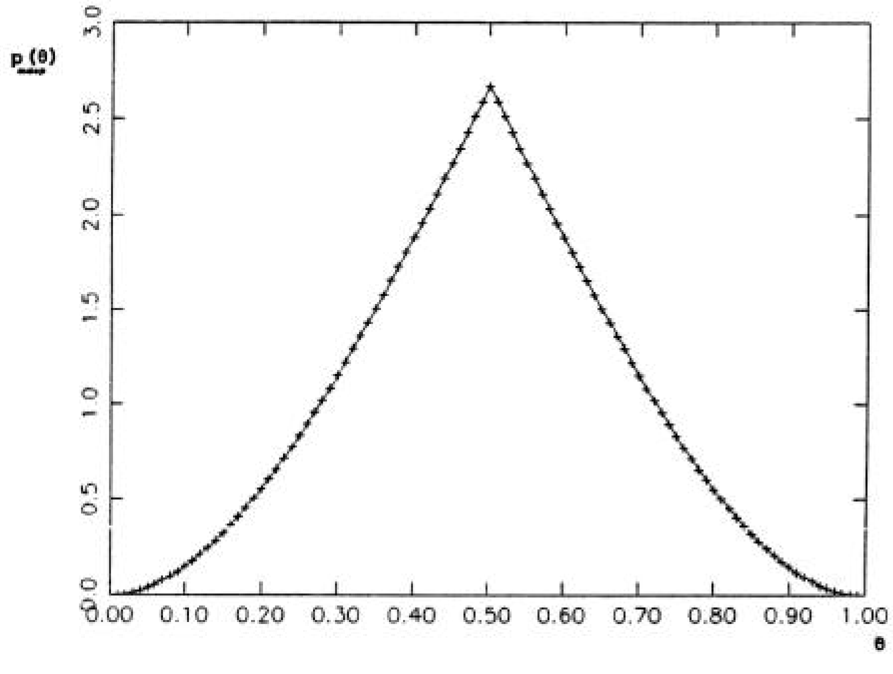

Fig. 8. Belief function of the independent case for A and B in Example 3.

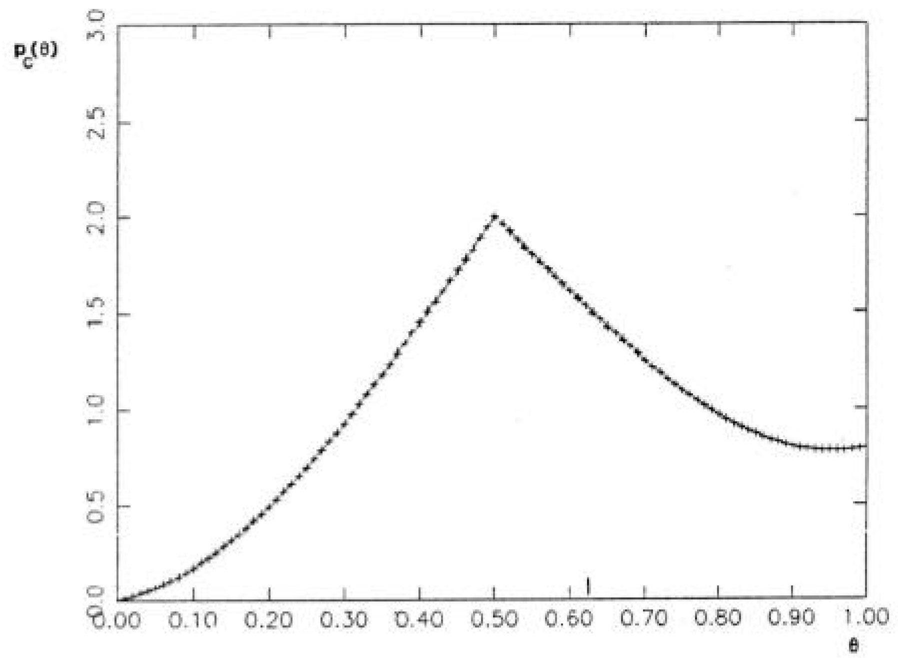

Fig. 9. Belief function of the hypothesis C in Example 3.

hypothesis C is plotted in Figure 9. Referring to Figures 6–9, the curve of the belief function $p_C(\theta)$ is dominated by the independent case of belief conjunction of the two given evidence, which is in Figure 8, and the right hand side of the graph indicates the perturbation provided by the B depending on A case, which is in Figure 6. Therefore, the probability density distribution of the belief function p_C meets what the intuition of human reasoning expects.

EXAMPLE 4. Given an inference rule R and evidence A and B, which are shown as an inference network in Figure 10, we are going to assess the belief function of the consequent supported by the two given evidence. Suppose that the node combining A and B is an OR node, which will constitute an intermediate evidence. The rule is

$$R\text{: if } (A\ OR\ B) \quad \text{then } C,$$

where the belief functions of evidence A and B, and the mutual dependency relationships between the evidence are the same as those given in Example 3, and the belief function of the rule R is

$$p_R(\theta) = 3 \cdot \theta^2,$$

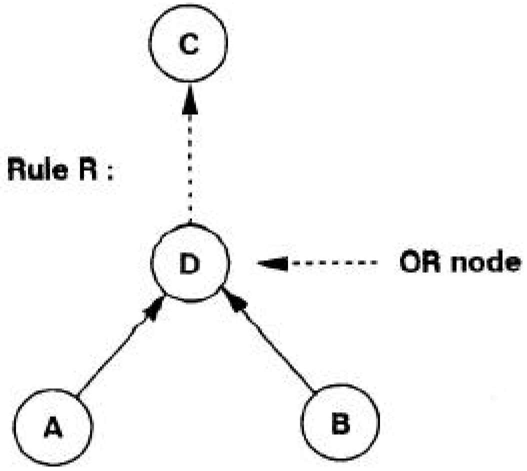

Fig. 10. The inference network of Example 4.

which is shown in Figure 11. We also assume that the rule R is independent of its antecedent. In the following, we will derive the belief function of the consequent C by our model.

By the result of the previous example, we can draw a conclusion that the belief function of the intermediate evidence D, which is an OR node, is

$$p_D(\theta) = p_A(\theta)+p_B(\theta)-p_{A\cap B}(\theta) = \begin{cases} 2-\frac{4}{5}\theta-\frac{48}{5}\theta^2+\frac{32}{5}\theta^3, & 0 \leqslant \theta \leqslant \frac{1}{2}, \\ \frac{-6}{5}-\frac{4}{5}\theta+\frac{48}{5}\theta^2-\frac{32}{5}\theta^3, & \frac{1}{2} \leqslant \theta \leqslant 1, \end{cases}$$

which is shown in Figure 12. Since the rule R is independent on any antecedent, the dependency relationship between D and R must be

$$\rho_{DR} = 0, \qquad \rho_{RD} = 0, \qquad \rho_{\text{indep}} = 1.$$

Here, for the sake of simplicity, we rewrite the belief function p_D as the following

$$p_D(\theta) = p_A(\theta)+p_B(\theta)-p_{A\cap B}(\theta) = \begin{cases} F_1(\theta), & 0 \leqslant \theta \leqslant \frac{1}{2} \\ F_2(\theta), & \frac{1}{2} \leqslant \theta \leqslant 1. \end{cases}$$

Then, according to the theory of belief conjunction given in Section 2.2, we can get

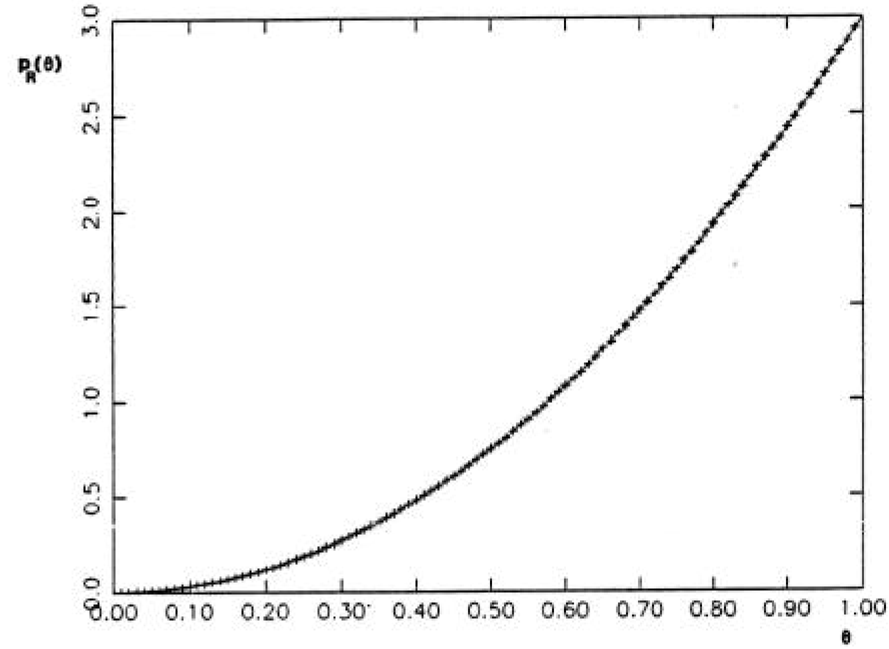

Fig. 11. Belief function of the rule R in Example 4.

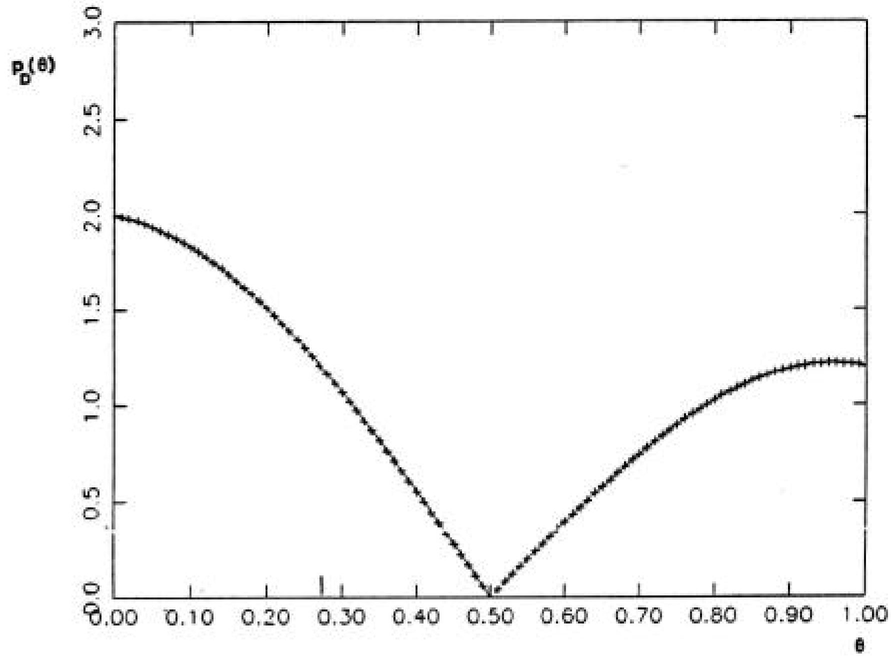

Fig. 12. Belief function of the intermediate evidence D in Example 4.

the following equations,

$$p_C(\omega) = \begin{cases} \int_0^{\omega} F_1(\mu)p_R(\omega-\mu)\,\mathrm{d}\mu, & 0 \leqslant \omega \leqslant \frac{1}{2}, \\ \int_0^{1/2} F_1(\mu)p_R(\omega-\mu)\,\mathrm{d}\mu + \int_{1/2}^{\omega} F_2(\mu)p_R(\omega-\mu)\,\mathrm{d}\mu, & \frac{1}{2} \leqslant \omega \leqslant 1, \\ \int_{\omega-1}^{1/2} F_1(\mu)p_R(\omega-\mu)\,\mathrm{d}\mu + \int_{1/2}^{1} F_2(\mu)p_R(\omega-\mu)\,\mathrm{d}\mu, & 1 \leqslant \omega \leqslant \frac{3}{2}, \\ \int_{\omega-1}^{1} F_2(\mu)p_R(\omega-\mu)\,\mathrm{d}\mu, & \frac{3}{2} \leqslant \omega \leqslant 2, \end{cases}$$

which can be further derived to be

$$p_C(\omega) = \begin{cases} \frac{1}{25}(8\omega^6 - 24\omega^5 - 5\omega^4 + 50\omega^3), & 0 \leqslant \omega \leqslant \frac{1}{2}, \\ \frac{1}{400}(1200\omega^2 - 432\omega + 56) - \\ \quad -\frac{1}{25}(8\omega^6 - 24\omega^5 + 5\omega^4 + 30\omega^3), & \frac{1}{2} \leqslant \omega \leqslant 1, \\ \frac{1}{400}(1200\omega^2 - 432\omega + 56) - \\ \quad -\frac{1}{25}(8\omega^6 - 24\omega^5 - 5\omega^4 - 110\omega^3 + \\ \quad +600\omega^2 - 590\omega + 140), & 1 \leqslant \omega \leqslant \frac{3}{2}, \\ \frac{1}{25}(8\omega^6 - 24\omega^5 + 5\omega^4 - 130\omega^3 + \\ \quad +600\omega^2 - 706\omega + 228), & \frac{3}{2} \leqslant \omega \leqslant 2. \end{cases}$$

Then we have to normalize the above equation so that the belief region is again [0,1].

$$p_C(\theta) = \begin{cases} \frac{1}{25}(1024\theta^6 - 1536\theta^5 - 160\theta^4 + 800\theta^3), & 0 \leqslant \theta \leqslant \frac{1}{4}, \\ \frac{1}{400}(9600\theta^2 - 1728\theta + 112) - \\ \quad -\frac{1}{25}(1024\theta^6 - 1536\theta^5 + 160\theta^4 + 480\theta^3), & \frac{1}{4} \leqslant \theta \leqslant \frac{1}{2}, \\ \frac{1}{400}(9600\theta^2 - 1728\theta + 112) - \\ -\frac{1}{25}(1024\theta^6 - 1536\theta^5 - 160\theta^4 - 1760\theta^3 + \\ \quad +4800\theta^2 - 2360\theta + 280), & \frac{1}{2} \leqslant \theta \leqslant \frac{3}{4}, \\ \frac{1}{25}(1024\theta^6 - 1536\theta^5 + 160\theta^4 - 2080\theta^3 + \\ \quad +4800\theta^2 - 2824\theta + 456), & \frac{3}{4} \leqslant \theta \leqslant 1, \end{cases}$$

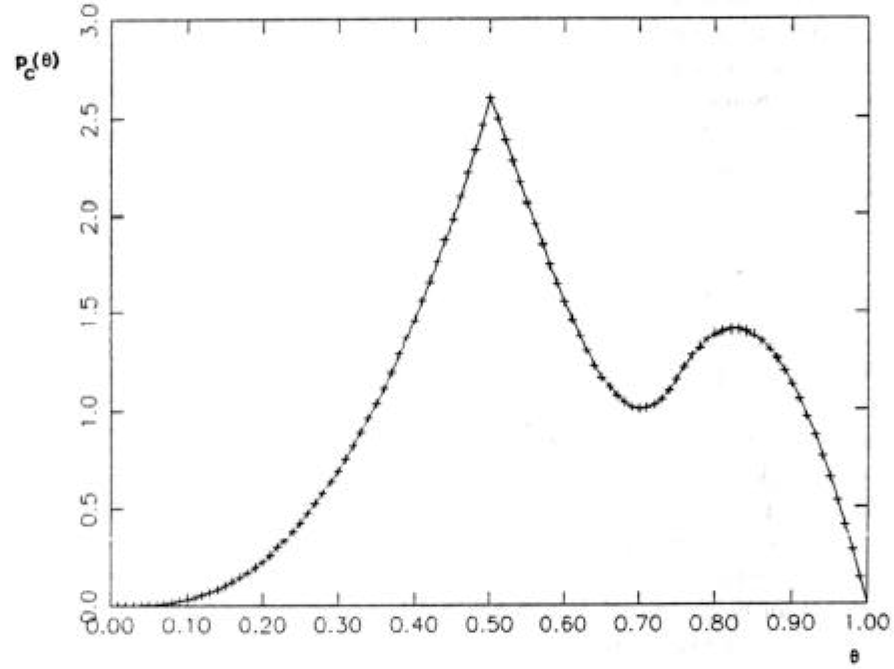

Fig. 13. Belief function of the consequent C in Example 4.

which is shown in Figure 12. We can also check the conservation property of the last equation,

$$\int_0^1 p_C(\theta)\,\mathrm{d}\theta = \frac{39}{1400} + \frac{1860}{5600} + \frac{2126}{5600} + \frac{729}{2800} = 1.$$

The overall propagation result has been derived and the property of probability conservation holds. The capability of the proposed model in managing the uncertainty aggregation of complicated continuous belief functions of evidence and inference rules in an inference network is illustrated in the above Example 4. If the conventional BPA approach is adopted in this example, no matter what thresholds are chosen to quantize the belief region, the fiedility of the original belief function will be seriously distorted or even totally lost. In turn, regardless of what kind of belief combination and belief propagation method is employed, it will lead to an unfaithful belief function of the final hypothesis.

4. Conclusion

Reasoning with uncertainty in a rule-based system is considered the aggregation of uncertain information about the validity of hypotheses from different sources. Section 1 discussed the three types of belief aggregation; belief conjunction, belief combination, and belief propagation. However, since the belief combination and the belief propagation are developed by the establishment of belief conjunction, they can be deemed as special cases of belief conjunction. If Dempster's rule [12] or Hau's approach [5] is adopted to perform the belief combination of the example provided in Section 2, the conflict resulting from the structural dependency of a lattice-structured inference network can not be satisfactorily resolved. In contrast, the proposed 2-D model solve this conflict resolution problem. The result of belief propagation must also depend on the mutual dependency relationship of the antecedent of the rule and the rule itself, which has long been ignored in the literature, in addition to the interpretation of the rule. The proposed model also embodies the inference in a lattice-structured inference network. Since an inference network can be treated as an AND/OR graph, the operation of any individual node (i.e., AND or OR) can reach the final result by applying the 2-D model. If the node is AND, then it is considered to be a *Conj* operation in the proposed model, while an OR node, then *Union* operation.

The new model offers several advantages over the prior attempts. First, the conflict in the dependency among the pieces of evidence in an inference network has been solved. Secondly, the discrete belief functions and the arbitrary continuous belief functions can be processed, which has not been researched to date. A continuous belief function is extremely advantageous because it can represent the vagueness of a human concept more accurately than the conventional means (i.e., BPA). Thirdly, the problem of dependency propagation of an intermediate evidence has been resolved. Finally, the conflict of belief propagation caused by the mutual dependency relationship of the antecedent of the rule and the rule itself has also been solved. This newly proposed strategy is intuitively closer to the human reasoning process than is Dempster's consensus seeking strategy [11, 12] or Hau's compromise seeking strategy [5], thus making this new model more successful and appealing than earlier attempts.

References

1. Cheng, Y. and Kashyap, R.L., A study of associative evidential Reasoning, *IEEE Trans. Pattern Anal. Machine Intell.* **11**(6) (June 1989).
2. Cohen, P.R., *Heuristic Reasoning about Uncertainty: An Artificial Intelligence Approach*, Morgan Kaufmann, Los Altos, CA., 1985.
3. Duda, R.O., Hart, P.E. and Nilsson, N., Subjective Baysian methods for rule-based inference systems, *Proc. AFIPS* (1976).
4. Ginsberg, M.L., Nonmonotonic reasoning using Dempster's rule, *Proc. National Conf. Artificial Intell.* (1984), 126–129.
5. Hau, H.Y. and Kashyap, R.L., Belief combination and propagation in a lattice-structured inference network, *IEEE Trans. Systems Man Cybernet.* **20**(1) (Jan./Feb. 1990).
6. Hummel, R.A. and Landy, M.S., A statistical viewpoint on the theory of evidence, *IEEE Trans. Pattern Anal. Machine Intell.* **PAMI-10**(2) (Mar. 1988).
7. Ihara, J., Extension of conditional probability and measures of belief and disbelief in hypothesis based on uncertain evidence, *IEEE Trans. Pattern Anal. Machine Intell.* **PAMI-9**(4) (Jul. 1987).
8. Ishizuka, M., Fu, K.S., and Yao, J.T.P., Inference procedure with uncertainty for problem reduction method, *Information Sci.* **28** (1983), 179–206.

9. Laskey, K.B. and Lehner, P.E., Assumptions, belief and probability, *Artificial Intell.* **41** (1989/90), 65–77.
10. Pearl, J., Fusion, propagation, and structuring in belief networks, *Artificial Intell.* **29** (1986), 241–288.
11. Shafer, G. and Logan, R., Implementing Dempster's rule for hierarchical evidence, *Artificial Intell.* **33** (1987).
12. Shafer, G. and Pearl, J., *Readings in Uncertain Reasoning*, Morgan Kaufmann, San Mateo, CA., 1990.
13. Shenoy, P.P. and Shafer, G., Propagating belief functions with local computation, *IEEE Expert*, (Fall 1986), 43–51.
14. Tahani, H. and Keller, J.M., Information fusion in computer vision using the fuzzy integral, *IEEE Trans. Systems Man Cybernet.* **20**(3) (May/June 1990).
15. Widman, L.E., Loparo, K.A. and Nielsen, N.R., *Artificial Intelligence, Simulation, and Modeling*, Wiley, New York, 1989.
16. Yen, J., Generalizing the Dempster–Shafer theory to fuzzy sets, *IEEE Trans. Systems Man Cybernet.* **20**(3) (May/June 1990).
17. Zadeh, L.A., A simple view of the Dempster–Shafer theory of evidence and its implication for the rule of combination, *Fuzzy Set Syst.* **11** (1983).